Lecturer: Thomas Rothvoss Due date: Friday, Nov 22, 2024, 11pm, on GradeScope

Problem Set 7

CSE 521 - Design and Analysis of Algorithms

Fall 2024

Exercise 1 (5+5+5+5=20pts)
For a vector $x \in \mathbb{R}^n$ we define $\text{supp}(x) := \{i \in [n] \mid x_i \neq 0\}$. Prove the following:

(i) Let $B \in \mathbb{R}^{m\times n}$ with $n > m$ and let $x \in \mathbb{R}^n$ be a vector with $x > \mathbf{0}$. Prove that there is a $y \in \mathbb{R}^n$ with $y \geq \mathbf{0}$, $|\text{supp}(y)| < n$ and $By = Bx$.
Hint: First prove that there is a vector $z \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ with $Bz = \mathbf{0}$ then use that vector z to update x.

(ii) Let $B \in \mathbb{R}^{m\times n}$ and let $x \in \mathbb{R}^n$ be a vector with $x \geq \mathbf{0}$ and $|\text{supp}(x)| > m$. Then there is a $y \in \mathbb{R}^n$ with $By = Bx$, $y \geq \mathbf{0}$ and $\text{supp}(y) \subset \text{supp}(x)$ (and in particular $|\text{supp}(y)| < |\text{supp}(x)|$).
Hint: Suppose for symmetry reasons that $x_1,\dots,x_k > 0$ and $x_{k+1} = \ldots = x_n = 0$. First prove that for the matrix

$$\tilde{B} := \begin{pmatrix} B \\ e_{k+1} \\ \vdots \\ e_n \end{pmatrix}$$

there is a vector $z \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ so that $\tilde{B}z = \mathbf{0}$. Then use that vector to update x.

(iii) Consider the polyhedron $P := \{x \in \mathbb{R}^n \mid Ax = b, x \geq \mathbf{0}\}$ where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ and assume that $P \neq \emptyset$. Fix a point $y \in P$ that minimizes $|\text{supp}(y)|$. Prove that $|\text{supp}(y)| \leq m$.

(iv) Let $X := \{x_1,\dots,x_n\} \subseteq \mathbb{R}^m$. Prove that for any $y \in \text{conv}(X)$, there is a subset $X' \subseteq X$ with $|X'| \leq m+1$ so that $y \in \text{conv}(X')$. Here

$$\text{conv}(X) := \Big\{ \sum_{i=1}^n \lambda_i x_i \mid \lambda_1,\dots,\lambda_n \geq 0 \text{ and } \sum_{i=1}^n \lambda_i = 1 \Big\}$$

is the *convex hull* of X (which is also the unique smallest convex set that contains all points of X).