Lecturer: Thomas Rothvoss Due date: Friday, Nov 15, 2024, 11pm, on GradeScope

Problem Set 6

CSE 521 - Design and Analysis of Algorithms

Fall 2024

Exercise 1 (5+5=10pts)

(i) Consider the unit radius n-dimensional sphere $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$. Prove that it has a $1/4$-net of size $2^{O(n)}$, i.e., there exists a set $N \subset S^{n-1}$ such that $|N| \leq 2^{O(n)}$ and for each point $x \in S^{n-1}$, there exists a point $y \in N$ such that $\|x - y\|_2 < 1/4$.

Hint: Construct N by greedily adding points at distance at least $1/4$ from each other. In order to bound $|N|$, compare the volume of disjoint balls of a suitable radius around each point in N to the volume of a single large ball including all of them. Feel free to use that the volume of an n-dimensional ball of radius r is $\mathrm{Vol}_n(rB_2^n) = c_n \cdot r^n$ where c_n is a constant independent of r.

(ii) Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let N be a $1/4$-net of the unit n-dimensional sphere. Show that

$$\sigma_1(M) \leq 2 \max_{y \in N} |y^T M y|,$$

where $\sigma_1(M)$ is the largest singular value of M; it is also the largest eigenvalue of M in absolute value.

Hint. You may use the fact that for any symmetric matrix $M \in \mathbb{R}^{n\times n}$ and vectors $x, y \in \mathbb{R}^n$ one has $|x^T Mx| \leq |(x-y)^T Mx| + |y^T M(x-y)| + |y^T My|$.

Exercise 2 (5+5=10 points)

Let A be the adjacency matrix of the *Erdös-Renyi random graph* $G(n, 1/2)$, i.e., a random graph where there is an edge between every pair of vertices, independently, with probability $1/2$.

(i) Prove that for any $c > 0$ there is a $C > 0$ so that for any vector $y \in \mathbb{R}^n$ with $\|y\|_2 = 1$ one has

$$\Pr[|y^T(A - \mathbb{E}[A])y| \leq C\sqrt{n}] \geq 1 - 2^{-cn}.$$

Hint: You may use Hoeffding's inequality in the following form:

Let $r_1, \ldots, r_n \sim \{-1, 1\}$ be independent random variables, and let $a_1, \ldots, a_n \in \mathbb{R}$. Then for any $t \geq 0$,

$$\Pr\Big[\Big|\sum_{i=1}^n r_i a_i\Big| > t\Big] \leq 2\exp\Big(-\frac{t^2}{2\sum_{i=1}^n a_i^2}\Big).$$

(ii) Prove that for any $c > 0$ there is a $C > 0$ so that

$$\Pr[\|A - \mathbb{E}[A]\|_{\mathrm{op}} \leq C\sqrt{n}] \geq 1 - 2^{-cn}.$$

Remark: Note that proving the required claim using (i) may seem impossible because you would have to use a union bound over *infinitely* many bad events. However, if you use a *net* as suggested in the previous exercise, you can reduce the question to finitely many vectors.