Lecturer: Thomas Rothvoss Due date: Friday, Nov 8, 2024, 11pm, on GradeScope

Problem Set 5

# CSE 521 - Design and Analysis of Algorithms

Fall 2024

**Exercise 1 (3+7=10pts)**

(i) Let $M \in \mathbb{R}^{n\times n}$ be a (symmetric) PSD matrix. Show that for any $P \in \mathbb{R}^{m\times n}$, $PMP^T \succeq 0$.

(ii) Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix that has $k$ positive eigenvalues for some integer $k \geq 1$. Use Cauchy's interlacing theorem (below) to show that for any $P \in \mathbb{R}^{m\times n}$, $PMP^T$ has at most $k$ positive eigenvalue.

**Theorem 1** (Cauchy's Interlacing Theorem)**.** *Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalues $\lambda_n \leq \cdots \leq \lambda_1$. For a vector $v \in \mathbb{R}^n$, let $\beta_n \leq \cdots \leq \beta_1$ be the eigenvalues of $M + vv^T$. Then,*

$$\lambda_n \leq \beta_n \leq \lambda_{n-1} \leq \beta_{n-1} \leq \cdots \leq \lambda_1 \leq \beta_1.$$

**Exercise 2 (3+3+4=10pts)**
In this problem you can use the following theorem.

**Theorem 2** (Hanson-Wright inequality)**.** *Let $\sigma_1, \ldots, \sigma_n \sim \{-1,1\}$ be independent random variables (i.e., $\sigma_i = +1$ w.p. 1/2 and $\sigma_i = -1$ otherwise) and let $A \in \mathbb{R}^{n\times n}$. Then for all $t \geq 0$*

$$\Pr[|\sigma^T A\sigma - \mathbb{E}[\sigma^T A\sigma]| > t] \leq 2\exp\left(-c\min\left\{\frac{t^2}{\|A\|_F^2}, \frac{t}{\|A\|_{\mathrm{op}}}\right\}\right)$$

*where $c > 0$ is a universal constant (independent of $n$).*

(i) Show that for any matrix $A \in \mathbb{R}^{m\times n}$,

$$\|A^T A\|_F^2 \leq \|A\|_{\mathrm{op}}^2 \|A\|_F^2$$

(ii) Show that for a random vector $\sigma \sim \{-1,1\}^n$, and for any $A \in \mathbb{R}^{m\times n}$ one has $\mathbb{E}[\|A\sigma\|_2^2] = \|A\|_F^2$.

(iii) Show that for a random vector $\sigma \sim \{-1,1\}^n$, and for any $A \in \mathbb{R}^{m\times n}$ and $0 < \varepsilon < 1$, we have,

$$\Pr\left[\left|\frac{\|A\sigma\|_2^2}{\|A\|_F^2} - 1\right| > \varepsilon\right] \leq 2\exp\left(-c\frac{\varepsilon^2\|A\|_F^2}{\|A\|_{\mathrm{op}}^2}\right)$$