Problem Set 4

514 - Networks and Combinatorial Optimization

Autumn 2023

Exercise 4.1 (10pts)
Prove that in a matrix, the maximum number of non-zero entries with no two in the same line (=row or column), is equal to the minimum number of lines that include all nonzero entries.

Example: Consider the following matrix

\[ A = \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & 0 & 0 \end{pmatrix} \]

where * means any non-zero entry. Then the non-zero entries can be covered by two lines (2nd row and first column) and this is optimal. Also we can select at most 2 non-zero entries that have all rows and columns distinct — for example the two entries (1,1) and (2,2) on the diagonal.

Exercise 4.2 (10pts)
Let \( A = (A_1, \ldots, A_n) \) be a family of subsets of some finite set \( X \). Prove that \( A \) has an SDR if and only if

\[ \left| \bigcup_{i \in I} A_i \right| \geq |I| \]

for each subset \( I \subseteq \{1, \ldots, n\} \).

Remark: Recall that an SDR is an injective map \( \pi: [n] \rightarrow X \) with \( \pi(i) \in A_i \) for all \( i = 1, \ldots, n \).

Exercise 4.3 (10pts)
A matrix is called doubly-stochastic if it is nonnegative and each row sum and each column sum is equal to 1. A matrix is called a permutation matrix if each entry is 0 or 1 and each column and each row contains exactly one 1.

i) Show that for each doubly stochastic matrix \( A = (a_{ij})_{i,j=1,\ldots,n} \), there exists a permutation \( \pi \in S_n \) so that \( a_{i,\pi(i)} \neq 0 \) for all \( i = 1, \ldots, n \).

ii) Derive that each doubly stochastic matrix is a convex linear combination of permutation matrices.

Hint: Set up a bipartite graph and prove the claim using König’s Theorem.

Remark. All three exercises are taken from A. Schrijver’s lecture notes.