Problem Set 7

514 - Networks and Combinatorial Optimization

Autumn 2022

Exercise 4.15 (10pts)
Let $D = (V,A)$ be a directed graph and let $C$ be the collection of directed circuits in $D$. For each directed circuit $C$ in $D$, let $\chi^C$ be the incidence vector of $C$, that is $\chi^C : A \to \{0,1\}$ with $\chi^C(a) = 1$ if $C$ traverses $a$ and $\chi^C(a) = 0$ otherwise. Show that $f$ is a non-negative circulation if and only if there exists a function $\lambda : C \to \mathbb{R}_{\geq 0}$ so that $f = \sum_{C \in C} \lambda(C) \cdot \chi^C$. That is, the nonnegative circulations form the cone generated by $\{\chi^C \mid C \in C\}$.

Exercise (Not in Schrijver – 10pts)
First answer the following:

(i) Let $D' = (V,A')$ be a directed graph with capacities $c'$ and $s,t \in V$. We call an $s$-$t$ flow $g : A' \to \mathbb{R}_{\geq 0}$ elementary if there is a single $s$-$t$ path $P$ in $D'$ so that

$$g(a) = \begin{cases} \text{value}(g) & \text{if } a \in P \\ 0 & \text{if } a \notin P \end{cases}$$

Now let $f^*$ be a maximum $s$-$t$ flow in $D$ under $c'$. Prove that there is an elementary $s$-$t$ flow $g$ under $c'$ with $\text{value}(g) \geq \frac{1}{|A'|} \text{value}(f^*)$.

Now let $D = (V,A)$ be a directed graph with integral capacities $c : A \to \mathbb{Z}_{\geq 0}$, distinguished vertices $s,t \in V$ and for the sake of simplicity suppose that for each arc $a \in A$ one has $a^{-1} \notin A$. Recall that $D_f = (V,A_f)$ is the residual graph for an $s$-$t$ flow $f$. We can also define residual capacities $c_f(a) := c(a) - f(a)$ for $a \in A$ with $f(a) < c(a)$ and $c_f(a^{-1}) := f(a)$ for $a \in A$ with $f(a) > 0$.

(ii) Let $f$ be any flow under $c$ and let $f^*$ be a maximum value $s$-$t$ flow. Prove that the residual graph $D_f$ contains an $s$-$t$ path $P$ where $c_f(a) \geq \frac{1}{|A|} (\text{value}(f^*) - \text{value}(f))$ for all $a \in P$.

(iii) Consider the modification of the Ford-Fulkerson algorithm where in each iteration we pick an $s$-$t$ path $P$ that maximizes $\min\{c_f(a) : a \in P\}$ where $f$ is the current flow. Prove that this algorithm takes at most $O(|A| \ln(2 \text{value}(f^*)))$ many iterations.

Exercise (Not in Schrijver – 10pts)
Let $D = (V,A)$ be a directed graph with capacities $c(a) := 1$ for all $a \in A$. Let $s,t \in V$ and assume that $\delta^\text{out}(t) = \emptyset$. Let $\delta^\text{in}(t) = \{a_1, \ldots , a_m\}$ be the arcs incoming to $t$. Define $M = (X,I)$ with $X := \{a_1, \ldots , a_m\}$ and $I := \{a_i \in X : f(a_i) = 1\}$ if $f$ is an $s$-$t$ flow under $c$ in $D$. Prove that $M$ is a matroid!

Remark. The first exercise is taken from A. Schrijver’s lecture notes.