Problem Set 6

514 - Networks and Combinatorial Optimization

Autumn 2022

Exercise 4.5 (10pts)
Let $D = (V,A)$ be a directed graph and let $s,t \in V$. Let $f : A \to \mathbb{R}_{\geq 0}$ be an $s$-$t$ flow of value $\beta$. Show that there exists an $s$-$t$ flow $f' : A \to \mathbb{Z}_{\geq 0}$ of value $\lceil \beta \rceil$ so that $\lfloor f(a) \rfloor \leq f'(a) \leq \lceil f(a) \rceil$ for every $a \in A$.

Exercise 4.7(i) (10pts)
In the following graph $D = (V,A)$ (edges labelled with capacities $c(a)$), compute a maximum $s$-$t$ flow under $c$ and a minimum $s$-$t$ cut $\delta_{\text{out}}(U)$. What are their values? It suffices to state the final outcomes.

Exercise (Not in Schrijver – 10pts)
Let $D = (V,A)$ be a directed graph with two distinguished nodes $s,t \in V$. A set $U$ is called an $s$-$t$ vertex cut if $U \subseteq V \setminus \{s,t\}$ and every $s$-$t$ path intersects $U$. A collection of $s$-$t$ paths $P_1, \ldots, P_N$ is called internally vertex disjoint if they have no nodes in common other than $s$ and $t$. Prove the following using the MaxFlow=MinCut Theorem: Let $D = (V,A)$ be a directed graph with $s,t \in V$ so that $(s,t) \notin A$. Then the maximum number of internally vertex-disjoint $s$-$t$ paths equals the minimum $|U|$ where $U$ is an $s$-$t$ vertex cut.

**Hint:** Create an auxiliary graph and apply the MaxFlow=MinCut Theorem there!

**Remark.** Two exercises are taken from A. Schrijver’s lecture notes.