Lecturer: Thomas Rothvoss Due date: Saturday, Oct 22, 2022, 11pm, on GradeScope

Problem Set 3

514 - Networks and Combinatorial Optimization

Autumn 2022

Exercise (Not in Schrijver's notes — 10 pts)

(a) Consider $P := \{x \in \mathbb{R}^n \mid Ax = b, x \geq \mathbf{0}\}$ for $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Fix a point $y \in P$ that minimizes $|\text{supp}(y)|$ where $\text{supp}(y) := \{j \in \{1, \ldots, n\} \mid y_j \neq 0\}$. Prove that $|\text{supp}(y)| \leq m$.

(b) Let $X \subseteq \mathbb{R}^n$. For any $x \in \text{conv}(X)$, there is a subset $X' \subseteq X$ with $|X'| \leq n+1$ so that $x \in \text{conv}(X')$.

Exercise 2.20 (10pts)
Let $A \in \mathbb{R}^{m \times n}$ and let $b \in \mathbb{R}^m$ with $m \geq n+1$. Suppose that $Ax \leq b$ has no solution x. Prove that there are indices $i_0, \ldots, i_n$ so that the system $A_{i_0}^T x \leq b_{i_0}, \ldots, A_{i_n}^T x \leq b_{i_n}$ has no solution x.

Exercise 2.27 (10 pts)
Let $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Let $\tilde{x}$ be a feasible solution of $\max\{c^T x \mid Ax \leq b\}$ and let $\tilde{y}$ be a feasible solution to $\min\{y^T b \mid y \geq \mathbf{0}; y^T A = c^T\}$. Prove that $\tilde{x}$ and $\tilde{y}$ are optimum solutions to the maximum and minimum, respectively if and only if for each $i = 1, \ldots, m$ one has: $\tilde{y}_i = 0$ or $A_i^T \tilde{x} = b_i$.

Remark. The last two exercises are taken from A. Schrijver's 2009 lecture notes.