Problem Set 2

514 - Networks and Combinatorial Optimization

Autumn 2022

Exercise 10.1 (10 pts)
Let \((X, \mathcal{I})\) be a pair with \(X\) finite and \(\mathcal{I} \subseteq 2^X\). Consider the following properties:

(i) \(\emptyset \in \mathcal{I}\)

(ii) If \(Y \in \mathcal{I}\) and \(Z \subseteq Y\), then \(Z \in \mathcal{I}\).

(iii) If \(Y, Z \in \mathcal{I}\) and \(|Y| < |Z|\), then \(Y \cup \{x\} \in \mathcal{I}\) for some \(x \in Z \setminus Y\).

(iv) For any \(Y \subseteq X\), any two bases of \(Y\) have the same cardinality.

Assume that (i)+(ii) hold. Prove that (iii) and (iv) are equivalent.

Exercise 10.2 (10 pts)
Let \(M = (X, \mathcal{I})\) be a matroid. An inclusion-wise minimal dependent set \(Y \subseteq X\) is called a circuit. Two elements \(x, y \in X\) are called parallel if \(\{x, y\}\) is a circuit. Show that if \(x\) and \(y\) are parallel and \(Y \in \mathcal{I}\) with \(x \in Y\), then \((Y \setminus \{x\}) \cup \{y\} \in \mathcal{I}\).

Exercise 2.1 (10 pts)
Let \(C \subseteq \mathbb{R}^n\). Prove that \(C\) is a closed convex set if and only if there is a collection \(\mathcal{F}\) of closed affine halfspaces so that \(C = \bigcap_{H \in \mathcal{F}} H\).

Remark. All exercises are taken from A. Schrijver’s lecture notes.