Chapter 3

Matroid Intersection

This chapter is a reproduction of a section in Lex Schrijver’s lecture notes, with somewhat more details.

3.1 Introduction

In a previous chapter of this course, we learned what a matroid is. It is a pair \( M = (X, \mathcal{I}) \) where \( X \) is called the groundset and \( \mathcal{I} \) are subsets of \( X \) that are also called the independent sets. Additionally, the matroid has to satisfy the following three axioms:

1. Non-emptyness: \( \emptyset \in \mathcal{I} \)
2. Monotonicity: If \( Y \in \mathcal{I} \) and \( Z \subseteq Y \), then \( Z \in \mathcal{I} \)
3. Exchange property: If \( Y, Z \in \mathcal{I} \) with \( |Y| < |Z| \), then there is an \( x \in Z \setminus Y \) so that \( Y \cup \{x\} \in \mathcal{I} \)

Examples for matroids are:

- The set of forests in an undirected graph form a graphical matroid.
- If \( v_1, \ldots, v_n \) are vectors in a vector space, then \( M = ([n], \mathcal{I}) \) with \( \mathcal{I} = \{I \subseteq [n] | \{v_i\}_{i \in I} \text{ linearly independent}\} \) is a linear matroid.
- A partition matroid with ground set \( X \) can be obtained as follows: take any partition \( X = B_1 \cup \ldots \cup B_m \) and select numbers \( d_i \in [0, \ldots, |B_i|] \). Then \( M = (X, \mathcal{I}) \) with \( \mathcal{I} := \{I : |I \cap B_i| \leq d_i \text{ for all } i = 1, \ldots, m\} \) is a matroid.

We already learned that one can use the greedy algorithm to find a maximum weight independent set. In this chapter, we will see that a way more complex problem also can be solved in polynomial time:
CHAPTER 3. MATROID INTERSECTION

Matroid Intersection

Input: Matroid \( M_1 = (X, \mathcal{I}_1) \), \( M_2 = (X, \mathcal{I}_2) \) on the same groundset

Goal: Find \( \max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\} \)

To understand that this is a non-trivial problem, we want to argue that it contains maximum bipartite matching as a special case. To see this, take any bipartite graph \( G = (V, E) \). Suppose that \( V = U \cup W \) with \( U = \{u_1, \ldots, u_{|U|}\} \) and \( W = \{w_1, \ldots, w_{|W|}\} \) are both sides. Then we can define two matroids that both have the edge set \( E \) as ground set as follows: take \( M_1 = (E, I_1) \) as the partition matroid with partitions \( \delta(u_1), \ldots, \delta(u_{|U|}) \), all with parameter \( d_i := 1 \). Similarly, we introduce \( M_2 = (E, I_2) \) as partition matroid with partitions \( \delta(w_1), \ldots, \delta(w_{|W|}) \). Now the matroid intersection problem asks to select as many edges as possible, where in each neighborhood \( \delta(u_i) \) and \( \delta(w_j) \) we select at most one edge. This is exactly maximum bipartite matching. See the figure below for an example:

3.2 The exchange lemma

For example if we have two spanning trees \( T_1, T_2 \) in a graph, then the exchange property implies that for any \( e \in T_1 \), there exists some edge \( f(e) \in T_2 \) so that \( (T_1 \setminus \{e\}) \cup f(e) \) is again a spanning tree. Now we will see that a stronger property is true: the map \( f : T_1 \rightarrow T_2 \) can be chosen to be bijective.

Lemma 3.1. Let \( M = (X, \mathcal{I}) \) be a matroid and let \( Y, Z \in \mathcal{I} \) be disjoint independent sets of the same size. Define a bipartite exchange graph \( H = (Y \cup Z, E) \) with \( E = \{(y, z) : (Y \setminus y) \cup z \in \mathcal{I}\} \). Then \( H \) contains a perfect matching.

Proof. Suppose for the sake of contradiction that \( H \) has no perfect matching. From Hall’s condition we know that there must be subsets \( S \subseteq Y \) and \( S' \subseteq Z \) so that all edges incident to \( S' \) must have their partner in \( S \) and \( |S| < |S'| \).
3.3. THE RANK FUNCTION

Since $|S| < |S'|$ and $S, S'$ are both independent sets, there is an element $z \in S'$ so that $S \cup \{z\} \in \mathcal{I}$. We can keep adding elements from $Y$ to $S \cup \{z\}$ until we get a set $U \subseteq Y \cup \{z\}$ with $|U| = |Y|$.

There is exactly one element in $Y \setminus U$; we call it $x$. Then $(Y \setminus x) \cup \{z\} = U \in \mathcal{I}$ and $(x, z) \in E$ would be an edge — a contradiction.

We will use that exchange graph more intensively later. Formally, for a matroid $M = (X, \mathcal{I})$ and an independent set $Y \in \mathcal{I}$, we can define $H(M, Y)$ as the bipartite graph with partitions $Y$ and $X \setminus Y$ where we have an edge between $y \in Y$ and $x \in X \setminus Y$ if

$$(Y \setminus y) \cup \{x\} \in \mathcal{I}.$$  

3.3 The rank function

Again, let $M = (X, \mathcal{I})$ be a matroid. Recall that an inclusionwise maximal independent set is called a basis. Moreover, all bases have the same size which is also called the rank of a matroid. One can generalize this to the rank function $r_M : 2^X \to \mathbb{Z}_{\geq 0}$ which is defined by

$$r_M(S) := \max\{|Y| : Y \subseteq S \text{ and } Y \in \mathcal{I}\}$$

which for a subset $S \subseteq X$ of the groundset, tells how many independent elements one can select from $S$.

Now suppose we have two matroids $M_1 = (X, \mathcal{I}_1)$ and $M_2 = (X, \mathcal{I}_2)$ over the same groundset. The rank function will be useful to decide at some point that we have found the largest joint independent set. Let us make the following observation:
**Lemma 3.2.** Let $M_1 = (X, \mathcal{I}_1)$, $M_2 = (X, \mathcal{I}_2)$ with rank functions $r_1$ and $r_2$. Then for any independent set $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$ and any set $U \subseteq X$ one has

$$|Y| \leq r_1(U) + r_2(X/U).$$

**Proof.** We have

$$|Y| = |U \cap Y| + |(X/U) \cap Y| \leq r_1(U) + r_2(X/U),$$

using that $Y$ is an independent set in both matroid.

Later in the algorithm, we will see that this inequality is tight for some $Y$ and $U$. As a side remark, for partition matroids in bipartite graphs, the lemma coincides with the fact that a vertex cover is always an upper bound to the size of any matching.

### 3.4 An reverse exchange lemma

We just saw that the exchange graph has a perfect matching between independent sets of the same size. We now show the converse, namely that a unique perfect matching between an independent set $Y$ and any set $Z$ implies that $Z$ is also independent. In the following, we will consider perfect matchings in the graph $H(M, Y)$ between $Y \Delta Z$. What we mean is a perfect matching $N$, matching nodes in $Y \setminus Z$ to nodes in $Z \setminus Y$ and each edge $(y, z) \in N$ satisfies $(Y \setminus y) \cup \{z\} \in \mathcal{I}$.

![Diagram of matching between Y and Z](image)

**Lemma 3.3.** Let $M = (X, \mathcal{I})$ be a matroid and let $Y \in \mathcal{I}$ be an independent set and let $Z \subseteq X$ be any set with $|Z| = |Y|$. Suppose that there exists a unique perfect matching $N$ in $H(M, Y)$ between $Y \Delta Z$. Then $Z \in \mathcal{I}$.

**Proof.** Let $E = \{(y, z) \in (Y \setminus Z) \times (Z \setminus Y) \mid (Y \setminus y) \cup \{z\} \in \mathcal{I}\}$ be all the exchange edges between $Y \setminus Z$ and $Z \setminus Y$.

**Claim:** $E$ has a leaf $y \in Y / Z$.

**Proof of claim:** By assumption there is a perfect matching $N \subseteq E$. Start at any node $w \in Y \Delta Z$. If you are on the “right side” $Z \setminus Y$, then move along a matching edge in $N$; if we are on the left hand side $Y \setminus Z$, take a non-matching edge. If we every revisit a node,

1. Recall that a leaf is a degree-1 node.
then we have found an even length path \(C \subseteq E\) that alternates between matching edges and non-matching edges. Hence \(N \Delta C\) is again a perfect matching, which contradicts the uniqueness. That implies that our path will not revisit a node, but that it will get stuck at some point. It cannot get stuck at a node in \(Z/Y\) because there is always a matching edge incident. Hence it can only get stuck at a node \(y \in Y/Z\) that is only incident to one edge \((y, z)\) and that edge must be in \(N\).

\[\]

Let \(z\) denote the element with \((y, z) \in N\). Note that \(Z' := (Z \setminus z) \cup \{y\}\) satisfies \(|Y \Delta Z'| = |Y \Delta Z| - 2\) and there is still exactly one perfect matching between \(Y \Delta Z'\) (which is \(N \setminus \{(y, z)\}\)). Hence we can apply induction and assume that \(Z' \in \mathcal{I}\).

\[\]

We know that \(r((Y \cup Z) \setminus y) \geq r((Y \setminus y) \cup \{z\}) = |Y|\). By the matroid exchange property, there is some element \(x \in (Y \cup Z) \setminus y\) so that \(S := (Z'/y) \cup \{x\}\) is an independent set of size \(|Y|\). If \(x = z\) then \(Z = S \in \mathcal{I}\) and we are done. Otherwise, \(x \in Y/Z\).

\[\]

As \(|S| > |Y \setminus y|\), there must be an exchange edge between \(y\) and a node in \(S/Y\). That contradicts the choice of \(y\).

\[\]

### 3.5 The algorithm

Now, suppose that we have two matroids \(M_1 = (X, \mathcal{I}_1)\) and \(M_2 = (X, \mathcal{I}_2)\) over the same ground set. Our algorithm starts with the independent set \(Y := \emptyset\) and then augments it iteratively. Suppose we already have some joint independent set \(Y \in \mathcal{I}_1 \cap \mathcal{I}_2\). We will
show how to either find another set \( Y' \in \mathcal{I}_1 \cap \mathcal{I}_2 \) with \(|Y'| = |Y| + 1\) or decide that \( Y \) is already optimal. Let us define sets
\[
X_1 := \{ y \in X \mid Y \cup \{ y \} \in \mathcal{I}_1 \} \quad \text{and} \quad X_2 := \{ y \in X \mid Y \cup \{ y \} \in \mathcal{I}_2 \}
\]
In other words, \( X_1 \) denotes the elements that could be added to the independent set \( Y \) so that we would still have an independent set in \( M_1 \). We define a directed graph \( H = (X, E) \) as follows: for all \( y \in Y \) and \( x \in X / Y \)
\[
(y, x) \in E \iff (Y / y) \cup \{ x \} \in \mathcal{I}_1
\]
\[
(x, y) \in E \iff (Y / y) \cup \{ x \} \in \mathcal{I}_2
\]
Let us check what this graph does for bipartite graphs (and \( M_1, M_2 \) are the partition matroids modelling both sides). In this case \( Y \) corresponds to a matching, \( X_1 \) are edges whose left-side node is unmatched by \( Y \) and \( X_2 \) are edges whose right-side node is unmatched. We also observe that a \( Y \)-augmenting path corresponds to a directed path in \( H \).

With a bit care, we can use the concept of augmenting paths also for general matroid.

**Lemma 3.4.** Suppose there exists a directed path \( z_0, y_1, z_1, \ldots, y_m, z_m \) starting at a vertex \( z_0 \in X_1 \) and ending at a node \( z_m \in X_2 \). If that is a shortest path, then
\[
Y' := (Y \setminus \{ y_1, \ldots, y_m \}) \cup \{ z_0, \ldots, z_m \} \in \mathcal{I}_1 \cap \mathcal{I}_2
\]

*Proof.* We will show that \( Y' \in \mathcal{I}_1 \), the other inclusion follows by symmetry. On the figure below, on the left hand side, we consider the directed path and on the right hand side, we consider only edges \( E \) of the exchange graph \( H(M_1, Y) \) that run between \( Y \setminus Z \) and \( Z \setminus Y \) for \( Z := (Y \setminus \{ y_1, \ldots, y_m \}) \cup \{ z_1, \ldots, z_m \} = Y' \setminus y_0 \).
Note that the edges \( \{(z_i, y_i) : i = 1, \ldots, m\} \) from the directed path form a perfect matching on \( Y \Delta Z \). While \( E \) may contain more edges than that, it does not contain a coord, which is an edge \((y_i, z_j)\) with \( j > i \). The reason is that in this case our \( X_1 - X_2 \) path would not have been the shortest possible one as we could have used the coord as shortcut. Now, consider the “complete” cordless graph \( E^* := \{(y_i, z_j) : i \geq j\} \). Then this graph does have only one perfect matching. In particular, \((y_1, z_1)\) has to be in a matching — then apply induction.

As the matching on \( Y \Delta Z \) is unique, by Lemma 3.3 we have \( Z = Y' / z_0 \in \mathcal{I}_1 \). We know that \( r_{M_1}(Y \cup Y') \geq r_{M_1}(Y \cup \{z_0\}) \geq |Y| + 1 \) since \( z_0 \in X_1 \) is one of the “\( M_1 \)-augmenting” elements. One the other hand \( r_{M_i}(Y \cup Y'/\{z_0\}) \leq |Y| \) as none of the other elements of \( Y' \) is in \( X_1 \) (here we use again that we have a shortest path). Hence, the only element that could possibly augment \( Y' / z_0 \) to an independent set of size \( |Y| + 1 \) is \( z_0 \) itself.

**Lemma 3.5.** Suppose there is no path from a node in \( X_1 \) to a node in \( X_2 \). Then \( Y \) is optimal. In particular we can find a subset \( U \subseteq X \) so that \( |Y| = r_{M_1}(U) + r_{M_2}(X \setminus U) \).

**Proof.** Let \( U := \{i \in X : \exists X_1 - i \text{ path in } H\} \) (or maybe more intuitively, \( X \setminus U \) are the nodes that are reachable from \( X_1 \)).

First, we claim that \( r_{M_1}(U) = |Y \cap U| \). One direction is easy: \( r_{M_1}(U) \geq r_{M_1}(U \cap Y) = |Y \cap U| \). For the other direction, suppose for the sake of contradiction that \( r_{M_1}(U) > |Y \cap U| \) and hence there is some \( x \in U \) so that \((Y \cap U) \cup \{x\}\) is an independent set of size \(|Y \cap U| + 1\). There are two case depending on whether or not \( x \) also increases the rank of \( Y \) itself:

- **Case** \( r_{M_1}(Y \cup \{x\}) = |Y| + 1 \). Then \( x \in X_1 \cap U \), which is a contradiction to the choice of \( U \).

- **Case:** \( r_{M_1}(Y \cup \{x\}) = |Y| \). Take a maximal independent set \( Z \) with \((Y \cup U) \cup \{x\} \subseteq Z \subseteq Y \cup \{x\}\). Then there is exactly one element \( y \in Y / U \), so that \( Z = (Y / y) \cup \{x\} \). This
implies that we have would contain a directed edge $(y, x)$. Then the node $x \in U$ is reachable from an element $y \notin U$, which contradicts the definition of $U$.

From the contradiction we obtain that indeed $r_{M_1}(U) = |Y \cap U|$. Similarly one can show that $r_{M_2}(X/U) = |Y \cap (X/U)|$ (which we skip for symmetry reasons). Overall, we have found a set $U$ so that $|Y| = |Y \cap U| + |Y \cap (X \setminus U)| = r_{M_1}(U) + r_{M_2}(X \setminus U)$.

It follows that:

**Theorem 3.6.** Matroid intersection can be solved in polynomial time.

*Proof.* Start from $Y := \emptyset$ and iteratively construct the directed exchange graph; compute shortest $X_1$-$X_2$ paths and augment $Y$ as long as possible.

The matroids that we have seen so far, all had some explicit representation. Note that the matroid intersection algorithm would work also in the black box model, where the only information that we have about the matroids is given by a so-called independence oracle. This is a method that receives a set $Y \subseteq X$ and simply answers whether or not this is an independent set.

Our algorithm provides a nice min-max formula for the size of joint independent sets:

**Theorem 3.7** (Edmond’s matroid intersection theorem). For any matroids $M_1 = (X, \mathcal{I}_1)$ and $M_2 = (X, \mathcal{I}_2)$ one has

$$\max\{|S| : S \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min_{U \subseteq X} \{r_{M_1}(U) + r_{M_2}(X \setminus U)\}$$

*Proof.* We saw the inequality “$\leq$” already in Lemma 3.2. When the matroid intersection algorithm terminates, then it has found a set $U$ providing equality.