Chapter 4

Flows

For a directed graph \( D = (V, A) \), we define \( \delta^+(U) := \{ (u, v) \in A : u \in U, v \notin U \} \) as the arcs leaving \( U \) and \( \delta^-(U) := \{ (u, v) \in A : u \notin U, v \in U \} \) as the arcs entering \( U \).

### 4.1 Circulations

In a directed graph \( D = (V, A) \), we call a function \( f : A \to \mathbb{R}_{\geq 0} \) a circulation if

\[
f(\delta^-(v)) = f(\delta^+(v)) \quad \forall v \in V.
\]

In some sense, circulations are flows that have no source and no sink. Let \( \ell : A \to \mathbb{R}_{\geq 0} \) be lower bounds and let \( u : A \to \mathbb{R}_{\geq 0} \) be upper bounds, then we say that a circulation \( f \) respects these bounds if \( \ell(a) \leq f(a) \leq u(a) \) for all \( a \in A \). Given such bounds, we can again define a residual graph \( D_f := (V, A_f) \) with \( A_f := \{ a \in A | f(a) < u(a) \} \cup \{ a^{-1} | a \in A : \ell(a) < f(a) \} \). Given such lower and upper bounds, it is not obvious which conditions should be satisfied so that a feasible circulation even exists. One obvious condition is that for every cut \( U \), the lower bound on the incoming flow should not exceed the upper bound on the outgoing flow. It turns out that this condition is also sufficient:

**Theorem 17** (Hoffman’s Circulation Theorem). Let \( D = (V, A) \) be a directed graph with \( \ell(a) \leq u(a) \forall a \in A \). Then there exists a circulation \( f \) with \( \ell(a) \leq f(a) \leq u(a) \) for all \( a \in A \) if and only if

\[
\ell(\delta^-(U)) \leq u(\delta^+(U)) \quad \forall U \subseteq V.
\]

**Proof.** “\( \Rightarrow \)”. For any cut \( U \) and any a circulation \( f \) with \( \ell(a) \leq f(a) \leq u(a) \) one has

\[
\ell(\delta^-(U)) \leq f(\delta^-(U)) \leq f(\delta^+(U)) \leq u(\delta^+(U)).
\]

“\( \Leftarrow \)”. For any function \( f \) (not necessarily a circulation), define \( \text{loss}_f(v) := f(\delta^+(v)) - f(\delta^-(v)) \). By compactness, there is a function \( f : A \to \mathbb{R}_{\geq 0} \) with \( \ell(a) \leq f(a) \leq u(a) \) for \( a \in A \) minimizing \( \sum_{v \in V} |\text{loss}_f(v)| \). Fix this choice of \( f \). Define

\[
S := \{ v \in V : \text{loss}_f(v) < 0 \} \quad \text{and} \quad T := \{ v \in V : \text{loss}_f(v) > 0 \}.
\]

Suppose that \( S \neq \emptyset \), since otherwise \( S = T = \emptyset \) and \( f \) is a circulation. If there is a path \( P \) in the residual graph \( D_f \) from a node in \( S \) to a node in \( T \), then \( f \) can be augmented along that
path decreasing $\|\text{loss}_f\|_1$. Hence we can assume that there is no $S$-$T$ path. Define $U := \{v \in V : v \text{ reachable from } S \text{ in } D_f\}$.

Then $A_f$ has no arcs leaving $U$. Hence for $a \in \delta^+(U)$ one has $f(a) = u(a)$ and $a \in \delta^-(U)$ one has $f(a) = \ell(a)$. Then

$$u(\delta^+(U)) - \ell(\delta^-(U)) = f(\delta^+(U)) - f(\delta^-(U)) \overset{\text{loss}_f(v) = 0 \forall v \in U \setminus S}{=} \text{loss}_f(S) < 0$$

That means $U$ is a cut violating the condition.

\[\square\]

### 4.2 Min Cost Circulations

In this section we will discuss an algorithm solving the following problem:

**Min Cost Circulation:** Given a directed graph $D = (V, A)$ with edge cost $c : A \to \mathbb{R}$, lower bounds $\ell : A \to \mathbb{R}_{\geq 0}$ and upper bounds $u : A \to \mathbb{R}_{\geq 0}$, find a circulation $f : A \to \mathbb{R}$ with $\ell(a) \leq f(a) \leq u(a)$ for all $a \in A$ minimizing $c(f) := \sum_{a \in A} c(a) \cdot f(a)$.

It is not difficult to model for example the maximum $s$-$t$ flow problem or the minimum cost max $s$-$t$ flow problem as a min cost circulation.

#### 4.2.1 The residual graph

Now that our circulation also includes cost, we need to extend the definition of a residual graph. For a directed $D = (V, A)$ with edge cost $c : A \to \mathbb{R}$, lower bounds $\ell : A \to \mathbb{R}_{\geq 0}$ and upper bounds $u : A \to \mathbb{R}_{\geq 0}$, we define the residual graph $D_f = (V, A_f)$ that contains the following two types of arcs:

- For $a \in A$ with $f(a) < u(a)$ we have $a \in A_f$ with residual capacity $u_f(a) := u(a) - f(a)$ and residual cost $c(a)$.
- For $a \in A$ with $f(a) > \ell(a)$ we have $a^{-1} \in A_f$ with residual capacity $u_f(a^{-1}) := f(a) - \ell(a)$ and residual cost $c(a^{-1}) := -c(a)$.

**Observation 18.** Let $f : A \to \mathbb{R}_{\geq 0}$ be a circulation with $\ell(a) \leq f(a) \leq u(a)$ and let $C \subseteq A_f$ be a directed circuit in the residual graph. Set $\lambda := \min\{u_f(a) : a \in C\}$. Then $f' : A \to \mathbb{R}$ with

$$f'(a) := \begin{cases} f(a) + \lambda & \text{if } a \in C \\ f(a) - \lambda & \text{if } a^{-1} \in C \\ f(a) & \text{otherwise} \end{cases}$$

is a circulation with $\ell(a) \leq f'(a) \leq u(a)$ and $c(f') = c(f) + \lambda \cdot c(C)$.
4.2.2 Flow decomposition

Let us call a circulation atomic if there is a single directed circuit $C \subseteq A$ so that

$$f(a) = \begin{cases} 
\lambda & \text{if } a \in C \\
0 & \text{otherwise}
\end{cases}$$

for some $\lambda \geq 0$.

The following technique is also called flow decomposition in the literature:

**Lemma 19.** Let $f : A \to \mathbb{R}_{\geq 0}$ be a circulation in $D = (V, A)$. Then there are atomic circulations $f_1, \ldots, f_k : A \to \mathbb{R}_{\geq 0}$ with $f = f_1 + \ldots + f_k$ and $k \leq |A|$.

**Proof.** Every node with positive in-degree in $\{a \in A \mid f(a) > 0\}$ has also positive out-degree. Find any simple cycle $C \subseteq \{a \in A \mid f(a) > 0\}$. Choose $\lambda := \min\{f(a) : a \in C\}$ and define an atomic circulation $f_1$ with circuit $C$ and value $\lambda$. Repeat with the non-negative circulation $f - f_1$. In each iteration the flow of at least one arc goes down to 0, which implies that at most $|A|$ iterations are needed.

4.2.3 Optimality criterion

It turns out that optimality of a circulation is equivalent to not having a negative cost cycle in the residual graph:

**Lemma 20.** Let $f$ be a circulation in $D = (V, A)$ with $\ell(a) \leq f(a) \leq u(a)$ for all $a \in A$. Then $f$ is not optimal $\iff$ there is a negative cost cycle in $D_f$.

**Proof.** “$\Rightarrow$” If $D_f$ contains a negative cost cycle $C$, then augmenting the flow on $f$ decreases the cost, see Observation 18.

“$\Leftarrow$”. Let $f^*$ be an optimum circulation. Define $f' : A \to \mathbb{R}_{\geq 0}$ by

$$f'(a) := \begin{cases} 
 f^*(a) - f(a) & \text{if } f(a) < f^*(a) \leq u(a) \\
 f(a^{-1}) - f^*(a^{-1}) & \text{if } f(a^{-1}) > f^*(a^{-1}) \geq \ell(a^{-1})
\end{cases}$$

Then $f'$ is a circulation with $c(f') = c(f^*) - c(f)$.

Using Lemma 19 we can split $f' = \sum_{i=1}^k f_i$ where $f_1, \ldots, f_k : A \to \mathbb{R}_{\geq 0}$ are atomic circulations. Then $\sum_{i=1}^k c(f_i) = c(f^*) - c(f) < 0$, hence for some $i$ we have $c(f_i) < 0$. The cycle corresponding to $f_i$ then has negative cost.
4.2.4 The algorithm

The obvious idea is to augment the current circulation along a cycle with negative cost in $D_f$. Already in case of maximum flows, we learned that the choice of a cycle should be done carefully in order to obtain a polynomial running time. It will turn out that the right cycle is the one that minimizes the average cost. Let us define

$$
\mu(f) := \min_{C \text{ cycle in } D_f} \left\{ \frac{c(C)}{|C|} \right\}
$$

as the cost of the minimum mean cycle. Note that we allow the empty cycle so that always $\mu(f) \geq 0$.

**Lemma 21.** Given a directed graph $D = (V, A)$ with edge cost $c : A \to \mathbb{R}$. Then a minimum mean length cycle can be found in time $O(|V|^2 \cdot |A|)$.

**Proof.** Consider the following slight variation of the Bellman-Ford algorithm: set

$$
d_0(u, v) := \begin{cases} 
0 & \text{if } u = v \\
\infty & \text{if } u \neq v
\end{cases}
$$

and

$$
d_k(u, v) := \min_{(w, v) \in A} \{ d_{k-1}(u, w) + c(w, v) \}
$$

for $k = 1, \ldots, |V|$ and $u, v \in V$. Then $d_k(u, v) = c(P)$ where $P$ is the $u$-$v$ walk with exactly $k$ arcs that minimizes $c(P)$. Computing all entries takes time $O(|V|^2 \cdot |A|)$. Then the cost of the minimum mean cycle is

$$
\min_{k=0, \ldots, |V|, u \in V} \left\{ \frac{d_k(u, u)}{k} \right\}
$$

(counting the entries for $k = 0$ as having value 0) and also the cycle itself can be recovered from the entries. \hfill \square

Note that the minimum mean cost cycle can be found with a more sophisticated argument in time $O(|V| \cdot |A|)$, see for example Chapter 7.3 in the textbook of Korte and Vygen.

The algorithm that will work for min cost circulation is the following:

<table>
<thead>
<tr>
<th>Minimum mean cycle canceling algorithm</th>
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<tbody>
<tr>
<td>(1) Compute any feasible circulation $f$ with $\ell(a) \leq f(a) \leq u(a)$ for all $a \in A$.</td>
</tr>
<tr>
<td>(2) WHILE $\exists$ cycle in $D_f$ with negative cost DO</td>
</tr>
<tr>
<td>(3) Compute minimum mean cycle $C \subseteq A_f$</td>
</tr>
<tr>
<td>(4) Augment $f$ along $C$ by $\min{uf(a) : a \in C}$</td>
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Step (1) can be implemented, by solving a min cost circulation problem in the graph $D' = (V, A')$ which for each $a = (u, v) \in A$ contains two parallel arcs $a', a''$ from $u$ to $v$ with parameters:

- $c(a') = -1$, $\ell(a') = 0$, $u(a') = \ell(a)$
- $c(a'') = 0$, $\ell(a'') = 0$, $u(a'') = u(a) - \ell(a)$

Then $f = 0$ is a feasible circulation in $D'$ and any mincost circulation in $D'$ corresponds exactly to a feasible circulation in $D$. 18
4.2.5 Node potentials

For a directed graph \( D = (V, A) \) with edge cost \( c : A \to \mathbb{R} \), a function \( \pi : V \to \mathbb{R} \) are called node potentials. These induce reduced costs \( c_{i,j}^\pi := c_{i,j} + \pi_i - \pi_j \).

**Lemma 22.** Let \( D = (V, A) \) be a directed graph with cost \( c : A \to \mathbb{R} \) and node potentials \( \pi \). Then any cycle \( C \subseteq A \) has \( c^\pi(C) = c(C) \).

*Proof.* Clear as the node potentials cancel out.

**Lemma 23.** Let \( D = (V, A) \) be a directed graph with arc cost \( c : A \to \mathbb{R} \). Then \( D \) has no negative cost cycle \( \iff \) there are node potentials with \( c_{i,j}^\pi := c_{i,j} + \pi_i - \pi_j \geq 0 \quad \forall (i,j) \in A \).

*Proof.* “\( \Rightarrow \)” Any cycle \( C \) has \( c(C) = c^\pi(C) \geq 0 \) by Lemma 22.

“\( \Leftarrow \)” We add an artificial node \( s \) and arcs \( (s,v) \) with \( c_{s,v}(v) := 0 \) for all \( v \in V \).

Let \( d(u,v) \) be the length of the shortest path \( u-v \) in the resulting graph. Note that \( d(u,v) \) maybe negative, but since there are no negative cost cycle, \( d \) is well defined. Define \( \pi(i) := d(s,i) \) as the node potentials. Then \( c_{i,j}^\pi = c_{i,j} + d(s,i) - d(s,j) \geq 0 \iff d(s,j) \leq d(s,i) + c_{i,j} \) which is the triangle inequality.

A circulation \( f : A \to \mathbb{R}_{\geq 0} \) is \( \varepsilon \)-optimal for \( \varepsilon \geq 0 \) if there are node potentials \( \pi : V \to \mathbb{R} \) so that \( c_{i,j}^\pi \geq -\varepsilon \) for all \( (i,j) \in A_f \). Let

\[
\varepsilon(f) := \min \{ \varepsilon \geq 0 : f \text{ is } \varepsilon\text{-optimal} \}
\]

The interpretation of this quantity is that \( \varepsilon(f) \) is the smallest amount that has to be added to the cost of the arcs in the residual graph to eliminate all negative cost cycles.

**Lemma 24.** For any circulation \( f \), \( \mu(f) = -\varepsilon(f) \).

*Proof.* “\( \geq \)” Let \( \pi \) be the node potentials valid for \( \varepsilon(f) \). The minimum mean cycle \( C \) has \( |C| \cdot \mu(f) = c(C) = c^\pi(C) \geq -\varepsilon(f) \cdot |C| \).

“\( \leq \)” Let use define a cost function \( \tilde{c}(u,v) := c(u,v) - \mu(f) \). Now there is no negative cost cycle w.r.t. \( \tilde{c} \) and there are node potentials \( \pi \) with \( \tilde{c}(i,j) + \pi_i - \pi_j \geq 0 \), which is the same as \( c(i,j) + \pi_i - \pi_j \geq \mu(f) \).

4.2.6 The main analysis

The analysis of the cycle cancelling method will consist of showing that the value of \( \varepsilon(f) \) is decreasing in the course of the algorithm.

**Lemma 25.** Update \( f \) to \( f' \) by augmenting along a minimum mean cost cycle. Then \( \varepsilon(f') \leq \varepsilon(f) \).
Proof. Abbreviate $\varepsilon := \varepsilon(f)$. Let $\pi : V \to \mathbb{R}$ be the node potentials with $c^\pi(a) \geq -\varepsilon$ for every arc $a \in A_f$. We will show that the same node potentials are still feasible for the updated graph $D_{f'}$.

Let $C \subseteq A_f$ be the minimum cost mean cycle. Note that its cost are $c^\pi(C) = -|C| \cdot \varepsilon$ by Lemma 24. That means that every arc $a \in C$ must have $c^\pi(a) = -\varepsilon$. The only new arcs $(i, j) \in A_{f'} \setminus A_f$ have $(j, i) \in C$. Hence the reduced cost are $c^\pi_{ij} = c_{ji} + \pi_i - \pi_j = -c^\pi_{ji} = \varepsilon \geq 0$.

We should remark that the claim of Lemma 25 is false if an arbitrary cycle is chosen for updates. To understand the final part of the analysis better, fix a current circulation $f$ with $\varepsilon := -\mu(f)$ and let $\pi$ be the corresponding potential. Suppose for the sake of argument that the next two mean cycles are $C_1$ and $C_2$ contains the reverse of the arc that was the bottleneck for $C_1$. Then the picture for the reduced cost in the residual graph will be as follows:

![Cycle C1 and C2](image)

But then the mean cost of $C_2$ has to be higher as it contains an arc with reduced cost $+\varepsilon$. This idea can be generalized as follows:

**Lemma 26.** Consider a sequence $\{f_i\}_{i \geq 0}$ of circulations where $f_{i+1}$ emerges from $f_i$ by augmenting along the minimum mean cycle in $D_{f_i}$. Then $\varepsilon(f_{|A|+1}) \leq (1 - \frac{1}{|V|}) \cdot \varepsilon(f_0)$.

Proof. Let $\varepsilon := \varepsilon(f_0)$ and fix the potentials $\pi : V \to \mathbb{R}$ with $c^\pi(a) \geq -\varepsilon$ for all $a \in A_{f_0}$.

**Claim I.** There is a $k \in \{0, \ldots, |A| + 1\}$ so that the minimum mean cost cycle $C \subseteq D_{f_k}$ in that iteration contains an arc $a \in C$ with $c^\pi(a) \geq 0$.

**Proof of claim.** As long as the current iteration $k$ uses a minimum mean cost cycle $C \subseteq D_{f_k}$ with $c^\pi(a) < 0$, every arc that appears new in the residual graph will non-negative reduced cost. Moreover, in every iteration at least one arc is bottleneck arc and will not appear in the residual graph of the next iteration. This can only go on for at most $|A|$ iterations.

**Claim II.** Consider the minimal such $k$ from Claim I. Then $\mu(f_k) \geq (1 - \frac{1}{|V|}) \cdot \varepsilon$.

**Proof of claim.** Let $C \subseteq A_{f_k}$ be the minimum mean cycle. We know that $c^\pi(a) \geq -\varepsilon \forall a \in A_{f_k}$. Then $c(C) = c^\pi(C) \geq (|C| - 1) \cdot (-\varepsilon)$ and hence $\mu(f_k) = \frac{c(C)}{|C|} \geq (1 - \frac{1}{|C|}) \cdot (-\varepsilon)$. The claim follows from $|C| \leq |V|$.

**Theorem 27.** Suppose the cost function is $c : A \to \{-c_{\text{max}}, \ldots, +c_{\text{max}}\}$. Then the minimum mean cycle cancelling algorithm terminates after $|V| \cdot (|A| + 1) \cdot \ln(2|V| \cdot c_{\text{max}})$ iterations.

Proof. Let $f$ be the flow before the 1st iteration and $f'$ is the circulation after $|V| \cdot (|A| + 1) \cdot \ln(2|V| \cdot c_{\text{max}})$ iterations. Then $f$ is $c_{\text{max}}$-optimal. Moreover,

$$\varepsilon(f') \leq c_{\text{max}} \cdot \left(1 - \frac{1}{|V|}\right)^{|V| \cdot \ln(2|V| \cdot c_{\text{max}})} \leq c_{\text{max}} \cdot \exp(- \ln(2|V| c_{\text{max}})) = \frac{1}{2|V|}.$$ 

Since the cost are integral, that implies that actually $\varepsilon(f') = 0$. 

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Surprisingly one can prove that the number of iterations is bounded by a polynomial only in \(|V|\) and \(|A|\), independent from cost and capacity values.

**Theorem 28** (Tardos 1985). The minimum mean cost cycle algorithm terminates after \(O(|A|^2|V| \ln(|V|))\) iterations.

We will not give a proof here.

**Exercises**

**Exercise 4.1.** Let \(D = (V, A)\) be a directed graph with incidence matrix \(M_D \in \{-1, 0, 1\}^{V \times A}\). For a set of arcs \(C \subseteq A\), we write \(\chi^C \in \{0, 1\}^A\) as the characteristic vector. Consider

\[
P = \{ x \in \mathbb{R}^A \mid M_D x = 0; \quad x \geq 0 \} \quad \text{and} \quad Q = \text{cone}\{\chi^C : C \subseteq A \text{ is a circuit}\}.
\]

Prove that \(P = Q\).

**Exercise 4.2.** Use Hoffman’s Circulation Theorem to prove the non-trivial direction of the MaxFlow=MinCut theorem using a suitable reduction. Recall that the non-trivial direction says that for a directed graph \(D = (V, A)\) with \(s, t \in V\), capacities \(u(a) \geq 0\) for \(a \in A\) and a maximum flow value of \(F\), there is an \(s\)-\(t\) cut of value \(F\).

**Exercise 4.3.** Let \(D = (V, A)\) be a directed graph with lower and upper bounds \(\ell, u : A \rightarrow \mathbb{R}_{\geq 0}\) and cost \(c : A \rightarrow \mathbb{R}\). Let \(f, f^*\) be circulations with \(c(f^*) < c(f)\). Show that there exists a cycle \(C \subseteq A_f\) in the residual graph with bottleneck capacity \(\alpha := \min\{u_f(a) : a \in C\}\) so that \(\alpha \cdot c(C) \leq \frac{1}{|A|} \cdot (c(f^*) - c(f))\).