

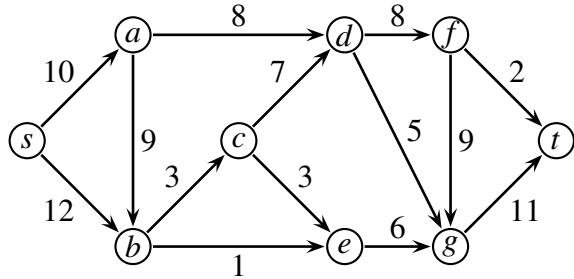
Problem Set 4

409 - Discrete Optimization

Winter 2026

Exercise 1 (6 points)

Consider the following network (G, u, s, t) (edges e are labelled with capacities $u(e)$):



- a) Run the Ford-Fulkerson algorithm to compute a maximum s - t flow. After each iteration draw the current flow f and the corresponding residual graph G_f . What is the optimum flow value?
- b) For the optimum flow f that you computed, define $S := \{v \in V \mid v \text{ is reachable from } s \text{ in } G_f\}$. Which are the nodes in S and what is the value $u(\delta^+(S))$ of the cut?

Exercise 2 (7 points)

Let (G, u, s, t) be a network with $G = (V, E)$ ¹ and let $f : E \rightarrow \mathbb{R}_{\geq 0}$ be a feasible s - t flow in that network. Moreover let f^* be a maximum s - t flow.

a) Let $S \subseteq V$ with $s \in S, t \notin S$. Prove that

$$\text{value}(f^*) \leq \text{value}(f) + \sum_{e \in \delta_{E_f}^+(S)} u_f(e)$$

where $\delta_{E_f}^+(S) := \{(i, j) \in E_f \mid i \in S, j \notin S\}$.

b) Let $\gamma > 0$ be the maximum value so that there is an s - t path P in G_f with $u_f(e) \geq \gamma$ for all $e \in P$. Prove that $\text{value}(f^*) \leq \text{value}(f) + \gamma m$ where $m := |E|$.

Exercise 3 (7 points)

Let (G, u, s, t) be a network with $n = |V|$ nodes and $m = |E|$ edges and $u(e) \in \mathbb{Z}_{\geq 0}$ for all $e \in E$. Suppose that f^* is a maximum s - t flow. Consider the following algorithm (which is a smarter version of Ford-Fulkerson):

¹As in the lecture notes we make the assumption that G does not contain both edges (i, j) and (j, i) to keep notation clean. Same assumption for exercise 3.

- (1) Set $f(e) := 0$ for all $e \in E$
- (2) WHILE $\exists s$ - t path in G_f DO

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- (3) Compute the s - t path P in G that maximizes $\gamma := \min\{u_f(e) \mid e \in P\}$
- (4) Augment f along P by γ

Note that in each iteration the algorithm chooses the path P that maximizes the *bottleneck capacity*. Let f_0, f_1, \dots, f_T be the sequence of flows computed by the algorithm where $0 = \text{value}(f_0) < \text{value}(f_1) < \dots < \text{value}(f_T) = \text{value}(f^*)$.

Hint: For (a) and (b) make use of the previous exercise.

- a) Prove that $\text{value}(f_1) \geq \frac{1}{m} \text{value}(f^*)$.
- b) Prove that $\text{value}(f_t) \geq \text{value}(f_{t-1}) + \frac{1}{m}(\text{value}(f^*) - \text{value}(f_{t-1}))$ for any $t \geq 1$.
- c) Prove that $\text{value}(f_t) \geq \text{value}(f^*) \cdot (1 - (1 - \frac{1}{m})^t)$ for any $t \geq 1$.
- d) Prove that the algorithm terminates after $T \leq \lceil m \cdot \ln(2 \cdot \text{value}(f^*)) \rceil$ iterations.