Problem Set 5

409 - Discrete Optimization

Winter 2025

Exercise 1 (8 points)

Consider the linear program

$$
\begin{aligned}
\max\ x_1 - 2x_2 \qquad & (P)\\
x_1 + x_2 \ &\leq\ 4\\
x_1 - 3x_2 \ &\leq\ 6\\
-2x_1 - x_2 \ &\leq\ -3\\
4x_1 - 3x_2 \ &\leq\ 15
\end{aligned}
$$

a) Draw the feasible region of LP (P).

b) Determine the optimum solution x^* by inspecting the picture.

c) State the dual program (to (P)).

d) Find the optimum solution to the dual. Check that it is feasible and the objective function value matches the objective function value for the dual.
   Hint: Your knowledge about duality will tell you very quickly how the dual solution has to look like. You do not need to run the simplex algorithm to find the dual solution!

Exercise 2 (5 points)

Let $x_1, \ldots, x_k \in \mathbb{R}^n$ be vectors and let $K := \mathrm{conv}\{x_1, \ldots, x_k\}$. Prove that if the origin $\mathbf{0} = (0, \ldots, 0)$ is not in K, then there is a vector $c \in \mathbb{R}^n$ with $c^T x_i \geq 1$ for all $i = 1, \ldots, k$.
Hint: Apply the hyperplane separation theorem.

Exercise 3 (7 points)

In this exercise, we want to study a system of a primal and dual LP that is in a different form than the one in the lecture. Consider

$$
\begin{array}{ll}
\max c^T x \qquad (P) \\
\quad Ax \ \leq\ b \\
\quad\ x \ \geq\ \mathbf{0}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ll}
\min b^T y \qquad (D) \\
\quad y^T A \ \geq\ c^T \\
\quad\ y \ \geq\ \mathbf{0}
\end{array}
$$

a) Give a direct proof that $(P) \leq (D)$. More concrete, suppose that $Ax \leq b, x \geq \mathbf{0}, y^T A \geq c^T, y \geq \mathbf{0}$ and then directly prove that $c^T x \leq y^T b$ without relying on any theorem from the lecture.

b) Show that the optimum values for (P) and (D) are the same, given that both systems are feasible.
   Hint: The easiest way to prove this is to rewrite (P) and (D) so that you arrive at systems $\max\{\tilde{c}^T \tilde{x} \mid \tilde{A}\tilde{x} \leq \tilde{b}\}$ and $\min\{\tilde{b}^T \tilde{y} \mid \tilde{y}^T \tilde{A} = \tilde{c}^T;\ y \geq \mathbf{0}\}$. Then you can use the result from the lecture that both systems have the same optimum value (again, given that both are feasible).