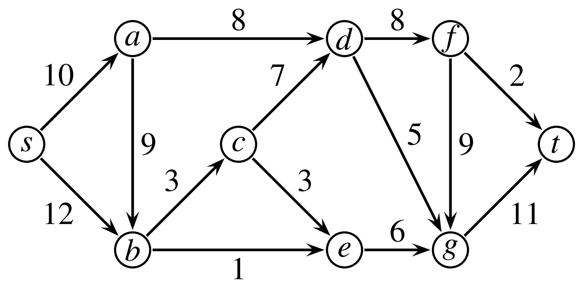
Lecturer: Thomas Rothvoss Due date: Friday, Feb 7, 2025, on GradeScope

Problem Set 4

# 409 - Discrete Optimization

Winter 2025

**Exercise 1 (6 points)**
Consider the following network $(G,u,s,t)$ (edges $e$ are labelled with capacities $u(e)$):




a) Run the Ford-Fulkerson algorithm to compute a maximum $s$-$t$ flow. After each iteration draw the current flow $f$ and the corresponding residual graph $G_f$. What is the optimum flow value?

b) For the optimum flow $f$ that you computed, define $S := \{v \in V \mid v \text{ is reachable from } s \text{ in } G_f\}$. Which are the nodes in $S$ and what is the value $u(\delta^+(S))$ of the cut?

**Exercise 2 (7 points)**
Let $(G,u,s,t)$ be a network with $G=(V,E)$[1] and let $f: E \to \mathbb{R}_{\geq 0}$ be a feasible $s$-$t$ flow in that network. Moreover let $f^*$ be a maximum $s$-$t$ flow.

a) Let $S \subseteq V$ with $s \in S, t \notin S$. Prove that

$$\text{value}(f^*) \leq \text{value}(f) + \sum_{e \in \delta^+_{E_f}(S)} u_f(e)$$

where $\delta^+_{E_f}(S) := \{(i,j) \in E_f \mid i \in S, j \notin S\}$.

b) Let $\gamma > 0$ be the maximum value so that there is an $s$-$t$ path $P$ in $G_f$ with $u_f(e) \geq \gamma$ for all $e \in P$. Prove that $\text{value}(f^*) \leq \text{value}(f) + \gamma m$ where $m := |E|$.

**Exercise 3 (7 points)**
Let $(G,u,s,t)$ be a network with $n = |V|$ nodes and $m = |E|$ edges and $u(e) \in \mathbb{Z}_{\geq 0}$ for all $e \in E$. Suppose that $f^*$ is a maximum $s$-$t$ flow. Consider the following algorithm (which is a smarter version of Ford-Fulkerson):

[1] As in the lecture notes we make the assumption that $G$ does not contain both edges $(i,j)$ and $(j,i)$ to keep notation clean. Same assumption for exercise 3.

(1) Set $f(e) := 0$ for all $e \in E$
(2) WHILE $\exists$ $s$-$t$ path in $G_f$ DO
    (3) Compute the $s$-$t$ path $P$ in $G$ that maximizes $\gamma := \min\{u_f(e) \mid e \in P\}$
    (4) Augment $f$ along $P$ by $\gamma$

Note that in each iteration the algorithm chooses the path $P$ that maximizes the *bottleneck capacity*. Let $f_0, f_1, \ldots, f_T$ be the sequence of flows computed by the algorithm where $0 = \text{value}(f_0) < \text{value}(f_1) < \ldots < \text{value}(f_T) = \text{value}(f^*)$.
**Hint:** For (a) and (b) make use of the previous exercise.

a) Prove that $\text{value}(f_1) \geq \frac{1}{m}\text{value}(f^*)$.

b) Prove that $\text{value}(f_t) \geq \text{value}(f_{t-1}) + \frac{1}{m}(\text{value}(f^*) - \text{value}(f_{t-1}))$ for any $t \geq 1$.

c) Prove that $\text{value}(f_t) \geq \text{value}(f^*) \cdot (1 - (1 - \frac{1}{m})^t)$ for any $t \geq 1$.

d) Prove that the algorithm terminates after $T \leq \lceil m \cdot \ln(2 \cdot \text{value}(f^*)) \rceil$ iterations.