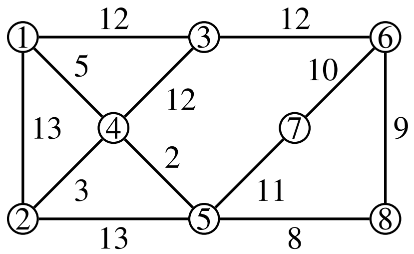
Lecturer: Thomas Rothvoss Due date: Friday, Jan 24, 2025, 11pm on GradeScope

Problem Set 2

409 - Discrete Optimization

Winter 2025

Exercise 1 (4 points)
Compute a minimum spanning tree in the graph $G = (V, E)$ depicted below. It suffices to give the final tree.



Exercise 2 (5 points)
Consider the following claim:

> *Let $G = (V, E)$ be an undirected graph with edge cost $c_e \in \mathbb{R}$ for all $e \in E$. If there are two minimum spanning trees $T_1, T_2 \subseteq E$, $T_1 \neq T_2$, then G contains two edges with the same cost.*

Is the claim true or false? If true, give a proof. If false, give a counterexample.

Exercise 3 (11 points)
Let $v_1, \dots, v_m \in \mathbb{R}^n$ be vectors. We assume that $\text{span}(v_1, \dots, v_m) = \mathbb{R}^n$. We call an index set $I \subseteq \{1, \dots, m\}$ a *basis*, if the vectors $\{v_i\}_{i \in I}$ are a basis of $\mathbb{R}^n$. We assume that we are given cost $c(1), \dots, c(m) \geq 0$ for all the vectors and abbreviate $c(I) := \sum_{i \in I} c(i)$ as the cost of a basis. We say that a basis $I^* \subseteq \{1, \dots, m\}$ is *optimal* if $c(I^*) \leq c(I)$ for any basis I.

i) Let $I, J \subseteq [m]$ be two different basis. Prove that for all $i \in I \setminus J$, there is an index $j \in J \setminus I$ so that $(I \setminus \{i\}) \cup \{j\}$ is a basis and also $(J \setminus \{j\}) \cup \{i\}$ is a basis.
Remark: If you are unsure how to prove this, you may want to lookup *Steinitz exchange lemma* from your Linear Algebra course. One variant that works is: Let $w_1, \dots, w_k \in \mathbb{R}^n$ be linearly independent and let $w = \sum_{\ell=1}^{k} \lambda_\ell w_\ell$ for $\lambda_1, \dots, \lambda_k \in \mathbb{R}$. Then for any index ℓ with $\lambda_\ell \neq 0$, also the vectors $w, w_1, \dots, w_{\ell-1}, w_{\ell+1}, \dots, w_k$ are linearly independent.

ii) Show that if a basis I is not optimal, then there is an *improving swap*, which means that there is a pair of indices $i \in I$ and $j \notin I$ so that $J := (I \setminus \{i\}) \cup \{j\}$ is a basis with $c(J) < c(I)$.
Remark: The proof of this claim is actually along the lines of Theorem 5 on page 17 in the lecture notes. I recommend to read that proof before.

iii) We want to compute an optimum basis and we want to use the following algorithm:

(1) Set $I := \emptyset$

(2) Sort the vectors so that $c(1) \leq c(2) \leq \ldots \leq c(m)$

(3) FOR $i = 1$ TO m DO

  (4) If the vectors $\{v_j\}_{j \in I \cup \{i\}}$ are linearly independent, then update $I := I \cup \{i\}$

Prove that the computed basis I is optimal.
Remark: Again, you might want to have a look into the correctness proof for Kruskal's algorithm in order to solve this.