Lecturer: Thomas Rothvoss Due date: Friday, Jan 17, 2025, 11pm, on GradeScope

Problem Set 1

409 - Discrete Optimization

Winter 2025

Exercise 1 (3 pts)
Let $G = (V, E)$ be any undirected graph. Recall that $\deg(v)$ gives the *degree* of $v \in V$ (which is the number of edges incident to v). Argue that

$$\sum_{v \in V} \deg(v) = 2 \cdot |E|.$$

Exercise 2 (8 pts)
Let $T = (V, E)$ be a graph that is a tree and that has $|V| = n$ nodes and assume that $n \geq 2$.

i) Show that T has at least 2 vertices of degree 1 (also called *leaves*).

ii) Use i) to prove by induction that the tree has exactly $|E| = n - 1$ many edges.
Remark: This quantity also falls out of another proof that we will see in the lecture. But please give your own proof by induction here.

iii) Show that T has at most $\frac{n}{2}$ many vertices that have degree 3 or higher.

Exercise 3 (9 pts)
In the lecture we saw that given a complete graph $K_n = (V, E)$ with edge cost $c_{ij} \geq 0$ for $\{i, j\} \in E$ one can compute a minimum cost TSP tour in time $O(2^n n^3)$ using dynamic programming. Here, we want to consider a variant of this problem:

> INPUT: Complete graph $K_n = (V, E)$ on n vertices with edge cost $c_{ij} \geq 0$ (for $\{i, j\} \in E$) and a parameter $m \in \{3, \ldots, n\}$.
> GOAL: Find a minimum cost cycle in K_n that connects exactly m nodes.

Give an algorithm (based on the dynamic program for TSP) that solves the problem. Which running time do you get? (a straightforward solution would get the minimum of $O(n^3 2^n)$ and $O(n^3 n^m)$ — the latter bound is better if m is a lot smaller than n).