Problem Set 6

409 - Discrete Optimization

Spring 2017

Exercise 1
Let $G = (V, E)$ be a bipartite graph with parts $V = V_1 \cup V_2$. Consider the linear program:

$$\begin{align*}
\min \sum_{u \in V} y_u \\
y_u + y_v &\geq 1 \quad \forall \{u, v\} \in E \\
y_u &\geq 0 \quad \forall u \in V
\end{align*}$$

a) If you write the problem in the matrix form $\min \{1^T y \mid Ay \geq 1^1; y \geq 0^0\}$, how is the matrix $A$ defined?

b) Prove that all extreme points of $P = \{y \in \mathbb{R}^V \mid Ay \geq 1^1; y \geq 0^0\}$ are integral ($A$ defined as in $a$)).

c) Which problem that you know from the lecture, does the above LP solve?

Exercise 2

a) Consider the triangle graph $G = (V, E)$ with 3 nodes and 3 edges

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 a
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 b
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and the matching polytope $P_M = \{x \in \mathbb{R}^E \mid \sum_{e \in \delta(v)} x_e \leq 1; x_e \geq 0\}$ associated with it. Write it in form $P_M = \{x \in \mathbb{R}^E \mid Ax \leq b\}$ and give the $6 \times 3$ constraint matrix $A$. What $3 \times 3$ submatrix $A_I$ of $A$ has $\det(A_I) \notin \{-1, 0, 1\}$? What is $\det(A_I)$? Compute $A_I^{-1}$. Which is the extreme point $x = A_I^{-1}b_I$ that belongs to this submatrix?

b) Which of the following matrices is TU? Argue why or why not!

$$A_1 = \begin{pmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
-1 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & -1 & 0
\end{pmatrix}.$$