Problem Set 4

409 - Discrete Optimization

Spring 2017

Exercise 1
Consider the following network with a directed graph $G = (V,E)$, capacities $u(e)$ (the labels of the edges), a source $s$ and a sink $t$ (assume that $M > 1$).

![Diagram of network with labels](image)

a) Argue that the Ford-Fulkerson algorithm with a poor choice of augmenting paths might take $M$ or more iterations.

b) Run the Edmonds-Karp algorithm on this network and give the flow in each iteration.

Exercise 2
In this exercise, you will give another proof of the Max-flow Min-Cut Theorem based on Hoffman’s Circulation Theorem.

Let $G = (V,E)$ be a directed graph. A circulation on $G$ is a function $f : E \rightarrow \mathbb{R}$ such that conservation of flow holds at every vertex $v \in V$. That is, a circulation must satisfy

$$\sum_{e \in \delta^+(v)} f(e) = \sum_{e \in \delta^-(v)} f(e)$$

for every vertex $v \in V$.

Hoffman’s Circulation Theorem states the following: Suppose $\ell : E \rightarrow \mathbb{R}$ and $u : E \rightarrow \mathbb{R}$ are functions that satisfy $\ell(e) \leq u(e)$ for every edge $e \in E$. Then there exists a circulation $f$ on $G$ satisfying

$$\ell(e) \leq f(e) \leq u(e)$$

for every edge $e \in E$ if and only if

$$\sum_{e \in \delta^-(A)} \ell(e) \leq \sum_{e \in \delta^+(A)} u(e)$$

for every set $A \subseteq V$.

Show that Hoffman’s Circulation Theorem implies the Max-flow Min-cut Theorem. To be precise, you should prove that given a network $(G = (V,E),c,s,t)$ ($c(e)$ giving the capacity on $e$), there exists
a flow of value equal to the minimum capacity $k$ of a cut in the network. You do not need to reprove the fact that the maximum value of a flow is at most the minimum capacity of a cut.

**Hint:** Let $G'$ be obtained from $G$ by adding a new edge $e_0 = (t, s)$. (It is possible that $e_0$ runs in parallel to an existing edge in $G$; this poses no problem.) Define functions $\ell, u : E \to \mathbb{R}$ by $\ell(e) = 0$ and $u(e) = c(e)$ for $e \in E$ and $\ell(e_0) = u(e_0) = k$. Now apply Hoffman’s Circulation Theorem to $G'$ to argue that the original network $G$ admits a flow of value $k$.

**Exercise 3**

Let $(G, u, s, t)$ be a network with $n = |V|$ nodes and $m = |E|$ edges and $u(e) \in \mathbb{Z}_{\geq 0}$ for all $e \in E$. Suppose that $f^*$ is the optimum max-flow. In this exercise, we want to develop a faster version of the Ford-Fulkerson algorithm. In fact, we want to modify the algorithm so that in each iteration the algorithm chooses the path $P$ that maximizes the bottleneck capacity $\gamma = \min\{u_f(e) \mid e \in P\}$. We call that algorithm “smart FF”.

a) Show that in the first iteration, smart FF finds already a flow $f$ with $\text{val}(f) \geq \frac{1}{2m} \text{val}(f^*)$.

**Hint:** The claim says essentially that even after we delete all edges $e$ that have small capacity, say $u(e) < \frac{1}{2m} \text{val}(f^*)$, the network will not become disconnected. It might be helpful to remember the MaxFlow=MinCut Theorem.

b) Now suppose we already computed some $s$-$t$ flow $f$. Show that there exists a flow $g$ in $G_f$ with $\text{val}(g) \geq \text{val}(f^*) - \text{val}(f)$.

**Hint:** This is somewhat the reverse process of augmenting a flow.

c) We want to generalize the claim in a). Consider any iteration of smart FF and say that $f$ is the flow that we computed so far. Show that there exists always a path $P$ in $G_f$ on which the bottleneck capacity $\min\{u_f(e) \mid e \in P\}$ is at least $\frac{1}{2m} (\text{val}(f^*) - \text{val}(f))$.

d) Show that smart FF needs at most $O(m \cdot \log(\text{val}(f^*)))$ many iterations. **Hint:** Argue that after $t$ iterations, our flow has a value of at least $\text{val}(f^*) \cdot (1 - (1 - \frac{1}{2m})^t)$. 