

## Problem Set 1

**409 - Discrete Optimization**

Spring 2017

**Exercise 1**

Let  $G = (V, E)$  be any undirected graph. Recall that  $\deg(v)$  gives the *degree* of  $v \in V$  (which is the number of edges incident to  $v$ ). Argue that

$$\sum_{v \in V} \deg(v) = 2 \cdot |E|.$$

**Exercise 2**

Let  $T = (V, E)$  be a graph that is a tree and that has  $|V| = n$  nodes and assume that  $n \geq 2$ .

- i) Show that  $T$  has at least 2 vertices of degree 1 (also called *leaves*).
- ii) Use i) to prove by induction that the tree has exactly  $|E| = n - 1$  many edges.  
**Remark:** This quantity also falls out of another proof that we have seen in the lecture. But please give your own proof by induction here.
- iii) Show that  $T$  has at most  $\frac{n}{2}$  many vertices that have degree 3 or higher.

**Exercise 3**

In the lecture we saw that given a complete graph  $K_n = (V, E)$  with edge cost  $c_{ij} \geq 0$  for  $\{i, j\} \in E$  one can compute a minimum cost TSP tour in time  $O(2^n n^3)$  using dynamic programming. Here, we want to consider a variant of this problem:

INPUT: Complete graph  $K_n = (V, E)$  on  $n$  vertices with edge cost  $c_{ij} \geq 0$  (for  $\{i, j\} \in E$ ) and a parameter  $m \in \{1, \dots, n\}$ .

GOAL: Find a minimum cost circuit in  $K_n$  that connects exactly  $m$  nodes.

Give an algorithm (based on the dynamic program for TSP) that solves the problem. Which running time do you get? (a straightforward solution would get the minimum of  $O(n^3 2^n)$  and  $O(n^3 n^m)$  — the latter bound is better if  $m$  is a lot smaller than  $n$ ).