VECTOR FUNCTIONS AND TANGENT LINES

Recall: Given a curve \( \mathbf{r}(t) = (x(t), y(t)) \) for \( t_{\text{start}} \leq t \leq t_{\text{end}} \).

The slope of the tangent line at \( t_0 \) is \( \frac{y'(t_0)}{x'(t_0)} \).

Imagine you “fly” through \( \mathbb{R}^2 \) and \( \mathbf{r}(t_0) \) is the position at time \( t_0 \). The direction of vector \( \mathbf{r}'(t_0) \) is the direction in which you “fly” at time \( t_0 \). In fact, if from time \( t_0 \) on you would fly straight and not change the speed, at time \( t_0 + 1 \) you would be at \( \mathbf{r}(t_0) + 1 \cdot \mathbf{r}'(t_0) \)
Example: Describe the path of

\[ \vec{r}(t) = (x(t), y(t)) = (3t, t^2 + 4) \quad \text{for} \quad -\infty < t < \infty \]

and eliminate the parameter \( t \).

We have \( x = 3t \iff t = \frac{x}{3} \).

Hence \( y = t^2 + 4 = \left(\frac{x}{3}\right)^2 + 4 \).

The curve is \((x, \frac{x^2}{9} + 4)\) for \(-\infty < x < \infty\), which is a parabola.
Effects of different curve parametrizations

Case 1: Curve $\vec{r}(t) = (t, \frac{t^2}{9} + 4)$; tangent vec.: $\vec{r}'(t) = (1, \frac{2}{9} t)$

Case 2: Curve $\vec{r}(t) = (3t, t^2 + 4)$. Tangent vec: $\vec{r}'(t) = (3, 2t)$

Case 3: Curve $\vec{r}(t) = (-3t, t^2 + 4)$; $\vec{r}'(t) = (-3, 2t)$
Paths in \( \mathbb{R}^3 \)

A path in \( \mathbb{R}^3 \) can be described with a vector function

\[
\vec{r}(t) = (x(t), y(t), z(t))
\]

The derivative \( \vec{r}'(t) = (x'(t), y'(t), z'(t)) \) now gives the direction vector for the line tangent to the point \( \vec{r}(t) \).

Example: For

\[
\vec{r}(t) = (3 \sin(t), 4 \cos(t), t) \quad \text{for} \quad t \geq 0
\]

find the parametric equations for the line tangent to the curve at \( t = \frac{\pi}{2} \).

Path starts at \( (0, 4, 0) \). \( z \) always increases.

\(-3 \leq x \leq 3 \) and \(-4 \leq y \leq 4 \) for all \( t \).
Example (cont.)

\[ \vec{r}(t) = (3 \sin(t), 4 \cos(t), t) \]
\[ \vec{r}(\frac{\pi}{2}) = (3, 0, \frac{\pi}{2}) \]
\[ \vec{r}'(t) = (3 \cos(t), -4 \sin(t), 1) \]
\[ \vec{r}'(\frac{\pi}{2}) = (0, -4, 1) \]

The tangent line is \{ \vec{r}(\frac{\pi}{2}) + t \vec{r}'(\frac{\pi}{2}) : t \in \mathbb{R} \}.

With parametric equations, the tangent line is

\[ x = 3, \, y = -4t, \, z = \frac{\pi}{2} + t \]
Def. The **unit tangent vector** to $\vec{r}(t)$ is

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

In the example above $\vec{r}'(t) = (3 \cos(t), -4 \sin(t), 1)$.

$$|\vec{r}'(t)| = \sqrt{9 \cos^2(t) + 16 \sin^2(t) + 1}$$

$$= \sqrt{9 \cos^2(t) + 9 \sin^2(t) + 7 \sin^2(t) + 1}$$

$$= \sqrt{7 \sin^2(t) + 10}$$

Hence

$$\vec{T}(t) = \left( \frac{3 \cos(t)}{\sqrt{7 \sin^2(t) + 10}}, \frac{-4 \sin(t)}{\sqrt{7 \sin^2(t) + 10}}, \frac{1}{\sqrt{7 \sin^2(t) + 10}} \right)$$

For example, $\vec{T}(\pi) = (-\frac{3}{\sqrt{10}}, 0, \frac{1}{\sqrt{10}})$. 