Given two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, their difference $\overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1)$ is called a vector.

Example:

$\overrightarrow{PQ} = (5 - 4, 4 - 2) = (1, 2)$

$\overrightarrow{AB} = (1 - 0, 2 - 0) = (1, 2)$

The vectors $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are equivalent.
Adding vectors

For \( \vec{a} = (a_1, a_2) \) and \( \vec{b} = (b_1, b_2) \) define
\[
\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2).
\]
Subtracting vectors

For $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ define $\vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2)$.

Example:

For $\vec{a} = (-2, 6)$, $\vec{b} = (3, 4)$ we have 
$\vec{a} - \vec{b} = (-2 - 3, 6 - 4) = (-5, 2)$
Scaling:
For a vector $\vec{a} = (a_1, a_2)$ and a scalar $c \in \mathbb{R}$, the scalar product is defined as $c\vec{a} = (ca_1, ca_2)$.

Example:

For $\vec{a} = (2, 3)$, we have $2\vec{a} = (4, 6)$ and $-2\vec{a} = (-4, -6)$
The magnitude of a vector \( \vec{a} = (x, y) \) is \( |\vec{a}| = \sqrt{x^2 + y^2} \)

The magnitude is the distance between the start point and the end point of the vector.

The direction of a vector \( \vec{a} \) is \( \frac{\vec{a}}{|\vec{a}|} \).

In \( \mathbb{R}^2 \) the zero vector is \( \vec{0} = (0, 0) \).

The zero vector has magnitude 0, but no direction.

Some observations:

- \( |\frac{1}{|\vec{a}|} \vec{a}| = 1 \)
- \( |c\vec{a}| = |c| \cdot |\vec{a}| \)
- If \( c > 0 \), then \( c\vec{a} \) and \( \vec{a} \) have same direction
- If \( c < 0 \), then \( c\vec{a} \) has the opposite direction to \( \vec{a} \)
Now everything in 3D

- For $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$, the vector from $P$ to $Q$ is $\overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.
- A vector $\vec{a} = (x, y, z)$ has magnitude $|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$.
- The zero vector is $\vec{0} = (0, 0, 0)$.
- Adding $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ gives $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$.
- etc..
THE STANDARD BASIS

In $\mathbb{R}^3$, the vectors $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, $\vec{k} = (0, 0, 1)$ are called the standard basis vectors.

Every vector in $\mathbb{R}^3$ can be written as a linear combination of the standard basis vectors.

Example:

$$(4, -2, 9) = (4, 0, 0) + (0, -2, 0) + (0, 0, 9)$$

$$= 4 (1, 0, 0) - 2 (0, 1, 0) + 9 (0, 0, 1)$$

$$= 4\vec{i} - 2\vec{j} + 9\vec{k}$$

Example: Given $\vec{a} = \vec{i} - \vec{j} + 2\vec{k}$ and $b = 3\vec{i} - \vec{k}$. How large is $|\vec{a} - 2\vec{b}|$?

We have $\vec{a} - 2\vec{b} = (1 - 2 \cdot 3)\vec{i} + (-1)\vec{j} + (2 + 2 \cdot 1)\vec{k} = (-5, -1, 4)$ and $|\vec{a} - 2\vec{b}| = \sqrt{(-5)^2 + (-1)^2 + 4^2} = \sqrt{42}$. 
Exercise (Spring 2013 Midterm 1, ex 4)
Let \( \ell \) be the line in \( \mathbb{R}^3 \) that passes though the points \((1, 2, 3)\) and \((4, 1, -1)\). Find the coordinates of the point where \( \ell \) intersects the \(xz\)-plane.

Solution: Let \( P = (1, 2, 3), \ Q = (4, 1, -1) \).
\[
\overrightarrow{PQ} = (4 - 1, 1 - 2, -1 - 3) = (3, -1, -4)
\]
The line \( \ell \) is
\[
\{P + t \overrightarrow{PQ} : t \in \mathbb{R} \} = \{(1, 2, 3) + t(3, -1, -4) : t \in \mathbb{R} \}
\]
Which point on it has the form \((x, 0, z)\)?
Calculate \(2 - t = 0 \Leftrightarrow t = 2.\)
The point is \((1, 2, 3) + 2(3, -1, -4) = (7, 0, -5)\)