DOUBLE INTEGRALS OVER GENERAL REGIONS (§15.3)

Example: What is the integral of \( f(x, y) = 2 - 3x + xy \) over the triangle \( R \) that is spanned by \((0, 0), (1, 0), (1, 3)\)?

Integration order 1:

\[
\int_0^1 \left( \int_0^{3x} 2 - 3x + xy \, dy \right) \, dx
\]

\[
= \int_0^1 \left[ 2y - 3xy + \frac{1}{2} xy^2 \right]_{y=0}^{3x} \, dx
\]

\[
= \int_0^1 6x - 9x^2 + \frac{9}{2} x^3 \, dx = \frac{9}{8}
\]

Integration order 2:

\[
\int_0^3 \left( \int_{y/3}^1 2 - 3x + xy \, dx \right) \, dy
\]

\[
= \int_0^3 \left( 2x - \frac{3}{2} x^2 + \frac{1}{2} x^2 y \right)_{x=y/3}^1 \, dy
\]

\[
= \int_0^3 \left( -\frac{1}{18} y^3 + \frac{1}{6} y^2 - \frac{1}{6} y + \frac{1}{2} \right) \, dy = \frac{9}{8}
\]
**Idea:** Choose the integration boundaries so that they represent the region.

**Case I:** Consider region of the form

\[ D = \left\{ (x, y) : \begin{array}{l} a \leq x \leq b; \\ g_1(x) \leq y \leq g_2(x) \end{array} \right\} \]

Then the signed volume under \( f \) on \( D \) is

\[
\iint_D f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx
\]

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**Case II:** Consider region of the form

\[ D = \left\{ (x, y) : \begin{array}{l} c \leq y \leq d \\ h_1(y) \leq x \leq h_2(y) \end{array} \right\} \]

Then the signed volume under \( f \) on \( D \) is

\[
\iint_D f(x, y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy
\]
Example (Final exam, Spring 2013)
Compute the double integral

\[
\int_0^{\sqrt{2}} \int_{y^2}^2 y^3 e^{x^3} \, dx \, dy
\]

**Question:** What is \( \int e^{x^3} \, dx \)?

**Answer:** No expression with basic functions exists.

**Solution:** Invert the integration order!

- **Step 1:** Make a picture of the region

  \[ D = \{(x, y) : y^2 \leq x \leq 2, \ 0 \leq y \leq \sqrt{2}\} \]

  ![Diagram of the region D]

- **Step 2:** Observe

  \[ D = \{(x, y) : 0 \leq x \leq 2, \ 0 \leq y \leq \sqrt{x}\} \]
Example (cont.)

• Step 3: Invert integration order and integrate

\[ \int_0^{\sqrt{2}} \int_{y^2}^2 y^3 e^{x^3} \, dx \, dy = \int_0^{\sqrt{2}} \left( \int_0^{\sqrt{x}} y^3 e^{x^3} \, dy \right) \, dx \]

\[ = \int_0^2 e^{x^3} \left( \frac{1}{4} \sqrt[4]{y^4} \right) \bigg|_{y=0}^{y=\sqrt{x}} \, dx \]

\[ = \frac{1}{4} \int_0^2 e^{x^3} x^2 \, dx \]

\[ = \frac{1}{12} \cdot e^{x^3} \bigg|_{x=0}^{x=2} = \frac{1}{12}(e^8 - 1) \]

(* ) Integration by substition: Choose \( f(u) := e^u \), \( g(x) = x^3 \). Then \( F(u) = \int e^u \, du = e^u \) and \( g'(x) = 3x^2 \). Hence

\[ \int \frac{1}{3} x^2 e^{x^3} \, dx = \int g'(x) \cdot f(g(x)) \, dx = F(g(x)) = e^{x^3} \]
Example (Final exam, Autumn 2011)

Sketch the region of integration and change the order of integration

\[ \int_{1}^{e} \int_{\ln(x)}^{-x+(1+e)} dy \, dx \]

Calculation:

(I) \( y = -x + (1 + e) \Leftrightarrow x = -y + 1 + e \)

(II) \( y = \ln(x) \Leftrightarrow x = e^y \)

Finally the integral with reversed int. order is

\[ \int_{0}^{1} \left( \int_{1}^{e^y} f(x, y) \, dx \right) \, dy + \int_{1}^{e} \left( \int_{1}^{1+e-y} f(x, y) \, dx \right) \, dy \]
Example (Midterm II, Aut. ’12, Loveless, Ex 3b)

Switching the order of integration, evaluate

\[
\int_0^2 \int_{\sqrt[4]{y}}^\frac{3}{2} \sqrt{x^4 + 1} \, dx \, dy
\]

Remark: \( \int \sqrt{x^4 + 1} \, dx \) has no closed formula with elementary functions.

\[
\int_0^2 \int_{\sqrt[4]{y}}^\frac{3}{2} \sqrt{x^4 + 1} \, dx \, dy = \int_0^2 \left( \int_0^{x^3} \sqrt{x^4 + 1} \, dy \right) \, dx
\]
\[
= \int_0^2 \sqrt{x^4 + 1} \, x^3 \, dx
\]
\[
= \frac{2}{12} (x^4 + 1)^{3/2} \bigg|_0^2 =
\]

Integration by substitution: \( f(u) := \sqrt{u} \), \( g(x) := x^4 + 1 \).

Then \( \int f(u) \, du = \frac{2}{3} u^{3/2} \) and \( g'(x) = 4x^3 \). Hence

\[
\int 4x^3 \sqrt{x^4 + 1} \, dx = \int g'(x) \cdot f(g(x)) \, dx = F(g(x)) = \frac{2}{3} (x^4 + 1)^{3/2}
\]
SUMMARY: INTEGRATION OVER REGIONS

For a region

\[ D = \left\{ (x, y) : a \leq x \leq b, \quad g_1(x) \leq y \leq g_2(x) \right\} = \left\{ (x, y) : c \leq y \leq d, \quad h_1(y) \leq x \leq h_2(y) \right\} \]

One has

\[
\iint_D f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy
\]

Effects of changing the integration order:

- “Difficulty” of integral may change dramatically
- Might need to split \( D \) into several regions

Hints: Better make a picture of \( D \)!