# Variation of the conformal radius

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**Abstract**

We study the change of the conformal radius $r(U)$ of a simply connected planar domain $U$ versus the subdomain $U_\epsilon$ consisting of the points of distance at least $\epsilon$ to $\partial U$. We show that the smallest exponent $\lambda$ such that $r(U) - r(U_\epsilon) = O(\epsilon^\lambda)$ satisfies $0.59 < \lambda < 0.91$. We also show that a well-known conjecture implies $\lambda = 2(\sqrt{2} - 1)$.

## 1 Statement of the results

Let $U$ be a simply connected proper subdomain of the plane and $z_0 \in U$. By the Riemann mapping theorem there exists a unique holomorphic bijection $\varphi$ from the unit disk $\mathbb{D}$ onto $U$ such that $\varphi(0) = z_0$, $\varphi'(0) > 0$ and we will call this map the Riemann map of $(U, z_0)$. The positive real number $\varphi'(0)$ is called the *conformal radius* $r(U, z_0)$, since $r(U, z_0)$ can also be characterized as the real number $r > 0$ such that there exists $\psi : U \to D(0, r)$ with $\psi(z_0) = 0$, $\psi'(z_0) = 1$. For $0 < \epsilon < d(z_0, \partial U)$ (the distance from $z_0$ to $\partial U$) we define $U_\epsilon$ as the connected component of $z_0$ of the set $\{z \in U;\ d(z, \partial U) > \epsilon\}$. It is again a simply connected domain containing $z_0$: the main goal of this paper is the study of the speed of convergence of $r(U_\epsilon, z_0)$ to $r(U, z_0)$ as $\epsilon \to 0$. Our first result is a simple consequence of the Koebe distortion theorem:

**Proposition 1.** *For $0 < \epsilon < d(z_0, \partial U)$ we have*

$$0 < r(U, z_0) - r(U_\epsilon, z_0) \leq 4\sqrt{r(U, z_0)}\sqrt{\epsilon} \tag{1.1}$$

**Proof:** By the Koebe theorem (see [4], p.9),

$$\frac{1}{4}(1 - |z|^2)|\varphi'(z)| \leq d(\varphi(z), \partial U) \leq (1 - |z|^2)|\varphi'(z)| \tag{1.2}$$

and

$$|\varphi'(z)| \geq \frac{1 - |z|}{(1 + |z|)^3}\varphi'(0) \tag{1.3}$$

for all $z \in \mathbb{D}$. From these two inequalities we deduce

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$$d(\varphi(z), \partial U) \geq \frac{1}{16}(1-|z|)^2\varphi'(0). \tag{1.4}$$

Let $D_\epsilon = \varphi^{-1}(U_\epsilon)$: inequality (1.4) implies that $D_\epsilon \supset D(0,r)$ if $\frac{1}{16}(1-r)^2\varphi'(0) \geq \epsilon$. We fix such a $r$ and let $\varphi_\epsilon$ be the Riemann map of $(U_\epsilon, z_0)$; we have $\varphi_\epsilon = \varphi \circ h_\epsilon$ where $h_\epsilon$ is the Riemann map of $D_\epsilon$. Since $\mathbb{D} \supset D_\epsilon \supset D(0,r)$, Schwarz' lemma implies that $1 > h'_\epsilon(z) \geq r$. The left inequality in (1.1) follows, as well as the inequality $r(U, z_0) - r(U_\epsilon, z_0) \leq (1-r)r(U, z_0)$. The right inequality in (1.1) follows by optimizing in $r$. □

In the sequel we will say that $U$ is normalized if $0 \in U, r(U,0) = 1$. We will also denote as usual by $S$ the set of injective holomorphic maps $\varphi : \mathbb{D} \to \mathbb{C}$ such that $\varphi(0) = 0,\ \varphi'(0) = 1$.

If $U$ is smooth, it is easy to see that $\sqrt{\epsilon}$ in (1.1) can be replaced by $C\epsilon$, where $C$ depends on the smoothness of $U$ only. The objective of this paper is to show that the "trivial bound" $\sqrt{\epsilon}$ in (1.1) can be improved for every simply connected domain $U$, but that it cannot be improved up to $\epsilon$. More precisely, we will prove the following

**Theorem 1.** *For all normalized simply connected domains $U$ there exists $\epsilon_0 > 0$ such that*

$$r(U,0) - r(U_\epsilon, 0) \leq \epsilon^{0.59} \quad \textit{for all } \epsilon < \epsilon_0. \tag{1.5}$$

*Conversely, there exists a normalized domain $U$ and $\epsilon_0 > 0$ such that*

$$r(U,0) - r(U_\epsilon, 0) \geq \epsilon^{0.91} \quad \textit{for all } \epsilon < \epsilon_0. \tag{1.6}$$

Furthermore we will establish, assuming the validity of a well-known conjecture about integral means (Kraetzer's conjecture, see section 4) the correct critical exponent. More precisely we will show that, assuming Kraetzer's conjecture is true and putting $\lambda = 2(\sqrt{2}-1) = 0.828..$, then (1.5) is true with 0.59 replaced by $\lambda - \alpha$ for any $\alpha > 0$, while for any $\delta > 0$ there exists a normalized $U$ such that (1.6) is true with 0.91 replaced by $\lambda + \alpha$.

In section 2 we reduce the problem to an area estimate. In section 3 we use lacunary series to construct a domain satisfying (1.6). In section 4 we show that the problem is equivalent to a certain estimate for integral means of derivatives of conformal maps, and use this connection to prove (1.5) and to determine the critical exponent (assuming Kraetzer's conjecture).

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## 2 Reduction of the problem

We consider a normalized domain $U$; we recall that $D_\epsilon = \varphi^{-1}(U_\epsilon)$. If $\Omega$ is a Borel subset of the plane we will denote by $|\Omega|$ its area (2-dimensional Lebesgue measure).

**Proposition 2.** *There exists a universal constant $C > 1$ such that*

$$\frac{1}{C}|\mathbb{D}\backslash D_\epsilon| \leq r(U,0) - r(U_\epsilon,0) \leq C|\mathbb{D}\backslash D_\epsilon|. \tag{2.1}$$

**Proof:**
a) Left inequality: We recall that $\varphi_\epsilon = \varphi \circ h_\epsilon$ where $\varphi_\epsilon, h_\epsilon$ are the Riemann maps of $(U_\epsilon, 0), (D_\epsilon, 0)$ respectively. Since $\varphi'(0) = 1$ we have $r(U,0) - r(U_\epsilon, 0) = 1 - h'_\epsilon(0) = 1 - a_1(\epsilon)$ if we write $h_\epsilon(z) = \sum_{n\geq 1} a_n(\epsilon) z^n$. Using $|D_\epsilon| = \pi \sum_{n\geq 1} n|a_n(\epsilon)|^2$, we obtain $2\pi(1 - a_1(\epsilon)) \geq \pi(1 - a_1(\epsilon)^2) = |\mathbb{D}| - |D_\epsilon| + \pi\sum_{n\geq 2} n|a_n(\epsilon)|^2 \geq |\mathbb{D}\backslash D_\epsilon|$, and the left inequality in (2.1) is proven with $C = 2\pi$.

b) Right inequality: We consider the usual dyadic decomposition of the disk: for each dyadic interval $I$ of length $2\pi 2^{-n}$ on the unit circle we consider the corresponding "square" $Q = \{z \in \mathbb{D}; z/|z| \in I, 1 - |z| \leq 2^{-n}\}$, and we denote $T(Q)$ the "top half" of $Q$ so that the $T(Q)'s$ form (modulo their boundaries) a partition of the disk. Notice also that each dyadic square $Q$ is the union of $T(Q)$ and two dyadic squares (the "children" of $Q$). For each $Q$ we also denote by $z_Q$ the middle of the top edge of $Q$. We then consider the usual random walk along the $z'_Q s$, choosing at each stage $z_Q$ one of the two children of $Q$. We stop the walk as soon as $(1 - |z_Q|^2)|\varphi'(z_Q)| \leq \epsilon'$ for some $\epsilon'$ which will be chosen in term of $\epsilon$ later. This provides us with a family of disjoint squares $(Q_j)$; fix such a square $Q_j$ and let $z \in T(Q)$ where $Q$ is the "father" of $Q_j$. By Koebe theorem and the maximality of $Q_j$ there exists a universal $C > 0$ such that $d(\varphi(z), \partial U) \geq C(1 - |z_Q|^2)|\varphi'(z_Q)| \geq C\epsilon' = \epsilon$ if $\epsilon' = \epsilon/C$. It follows that $\mathbb{D}\backslash D_\epsilon \subset \cup_j Q_j$ and, writing $D'_\epsilon = \mathbb{D}\backslash \cup_j Q_j$, we have $r(\mathbb{D}, 0) - r(D_\epsilon, 0) \leq r(\mathbb{D}, 0) - r(D'_\epsilon, 0)$. The proof will thus be complete if we can prove:

$$r(\mathbb{D},0) - r(D'_\epsilon,0) \leq C|\mathbb{D}\backslash D'_\epsilon|, \tag{2.2}$$

$$|\mathbb{D}\backslash D'_\epsilon| \leq C|\mathbb{D}\backslash D_\epsilon|. \tag{2.3}$$

We begin with (2.3). It suffices to show that for each $j$ there exists a square $Q \subset Q_j$ such that $|Q| \geq c|Q_j|$ and $Q \subset \mathbb{D}\backslash D_\epsilon$. To prove this we first use the fact [4] that there exists a geodesic $\gamma$ (for the hyperbolic metric) joining $z_{Q_j}$ to $Q_j \cap \partial\mathbb{D}$ and $C > 0$ such that the euclidean length of $\gamma$ is $\leq C(1 - |z_{Q_j}|)$. We then consider the first point $z$ on the geodesic such that $d(\varphi(z), \partial U) = \frac{\epsilon}{2}$. Between $z_{Q_j}$ and $z$ we thus have $C\epsilon \geq d(\varphi(\zeta), \partial U) \geq \frac{\epsilon}{2}$, from which it follows that the hyperbolic distance between $z_{Q_j}$ and $z$ is less than a (universal) $C > 0$. It thus suffices to choose a square centered at $z$ with side $\epsilon'(1-|z_Q|), \epsilon'$ small enough (independently of $j$).

Next we prove (2.2): we consider the increasing sequence of domains $\Omega_n$ with $\Omega_0 = D'_\epsilon$ and $\Omega_{n+1} = \Omega_n \cup Q_{j_n}$ where $Q_{j_n}$ is (one of) the largest square(s) in $\mathbb{D}\backslash\Omega_n$. Let $m = \omega(0, \partial Q_{j_n}, \Omega_n)$, the harmonic measure of $\partial Q_{j_n}$ from 0 in $\Omega_n$. Let $r_n \in ]0,1[$ be such that the top edge of $Q_{j_n}$ lies in the circle of center 0 and radius $r_n$. We have $\Omega_n \supset D(0, r_n)$ so, by the maximum principle,

$$\begin{aligned} C\text{diam}(Q_{j_n}) &\geq \omega(0, \partial Q_{j_n}, \mathbb{D}\backslash Q_{j_n}) \geq m \\ &\geq \omega(0, \partial Q_{j_n}, D(0, r_n)) \geq c \text{ diam}(Q_{j_n}) \end{aligned} \tag{2.4}$$

Let $\varphi_n$ be the Riemann map of $(\Omega_n, 0)$. By construction, the $\Omega'_n s$ are uniform quasidisks, so that the $\varphi_n$ have uniformly quasiconformal extensions to $\mathbb{C}$. Therefore $Q'_n = \varphi_{n+1}^{-1}(Q_{j_n})$ is a $K-$ quasidisk for some $K$ not depending on $n$, and there are points $\zeta_n \in \partial\mathbb{D}$ and $\rho_n \in ]0,1[$ with $D(\zeta_n, c\rho_n) \subset Q'_n \subset D(\zeta_n, C\rho_n)$. The maximum principle and (2.4) shows that $m \sim \rho$ and hence $|Q'_n| \sim |Q_{j_n}|$. Since $1 - r(\mathbb{D}\backslash D(1,\rho), 0) \sim \rho^2$, it follows that $1 - (\varphi_{j_{n+1}}^{-1} \circ \varphi_{j_n})'(0) \sim |Q'_n| \sim |Q_{j_n}|$ and $(\varphi_{n+1}^{-1} \circ \varphi_n)'(0) = \frac{\varphi'_n(0)}{\varphi'_{n+1}(0)}$, which implies that

$$\varphi'_{n+1}(0) - \varphi'_n(0) = \varphi'_{n+1}(0)(1 - \frac{\varphi'_n(0)}{\varphi'_{n+1}(0)}) \sim 1 - \frac{\varphi'_n(0)}{\varphi'_{n+1}(0)} \sim |Q|.$$

Finally $1 - \varphi'_0(0) = \sum_n (\varphi'_{j_{n+1}}(0) - \varphi'_{j_n}(0)) \sim \sum_n |Q_{j_n}| = \sum_j |Q_j|$, and (2.2) is proven. □

**Remark:** In the next section we will use this stopping time construction with a q-adic decomposition instead of the dyadic one. It is clear that the estimates of this section, in particular (2.2) and (2.3), remain valid.

# 3 Proof of (1.6)

We will use a standard construction of simply connected domains, based on the following univalence criterion due to Becker [1]:

**Theorem 2.** *If $\varphi$ is analytic in $\mathbb{D}$ and if*

$$(1-|z|^2)\left|z\frac{\varphi''(z)}{\varphi'(z)}\right| \leq k \leq 1 \tag{3.1}$$

*for all $z \in \mathbb{D}$ , then $\varphi$ is injective in $\mathbb{D}$; if moreover $k \in (0,1)$ in (3.1) then $\varphi$ has a quasiconformal extention to the plane.*

We define $\varphi$ by

$$\varphi(z)=\int_0^z e^{g(u)}du,\ g(z)=\alpha\sum_{n\geq 1} z^{q^n} \tag{3.2}$$

where $q$ is an integer $\geq 3$. We choose $\alpha, q$ according to the following result of Pommerenke ([4] p.189):

**Proposition 3.** *If*

$$B_q=\max_{0\leq x\leq 1}\{2\sum_{-\infty}^{+\infty} q^{n+x}\exp(-q^{n+x})\}$$

*then, if $\alpha<1/B_q$ we have (3.1) with some $k<1$.*

From now on we fix $\alpha<1/B_q$ ($q\geq 3$ will be chosen later) and consider the function $\varphi$ defined by (3.2). We use the following analogy between lacunary series and sums of independent random variables:

**Proposition 4.** *For every integer $q\geq 2$ there exists a constant $K(q)\geq 0$ such that $\forall t\in[0,2\pi[,\ \forall n\geq 0$,*

$$\left|\sum_{k=0}^{n} e^{iq^k t}-\sum_{k=0}^{\infty}[(1-q^{-n})e^{it}]^{q^k}\right|\leq K(q). \tag{3.3}$$

The functions playing the role of independent variables are the functions $X_k(t)=\cos(q^k t)$.We put $S_n=\alpha(X_1+..X_n)$ and define $S'_n$ as the function that has the constant value $\frac{1}{|I|}\int_I S_n$ on each dyadic interval $I$ of order $n$. The functions $S_n$ and $S'_n$ are essentially equal in the sense that $|S_n-S'_n|\leq C$ independently of $n$. For $z=(1-q^{-n})e^{it}$ we thus have $(1-|z|^2)|\varphi'(z)|\sim\exp(\alpha S'_n(t)-n\log q)$. Now consider the dyadic decomposition of the disc and let $T(Q)$ be the top $\frac{q-1}{q}$th of $Q$, so that $Q\backslash T(Q)$ consists of $q$ subcubes. Setting

$$\epsilon=e^{-N},$$

we replace the stopping time of the last section by

$$T_N=\inf\{n\geq 0;\ S'_n-n\log q\leq -N\}. \tag{3.4}$$

By the above arguments and (2.3) we have

$$|\mathbb{D}\backslash D_\epsilon|\geq cE(q^{-T_N}), \tag{3.5}$$

where $E$ stands for expectation, which is the integral over the circle. We define $M_n(\theta)=E(e^{\theta S_n})$.

**Lemma 3.1.** *For every $\theta \in \mathbb{R}$ there exists $C \geq 1$ such that*

$$\forall n, k \geq 1,\ \frac{1}{C} M_n(\theta) M_k(\theta) \leq M_{n+k}(\theta) \leq C M_n(\theta) M_k(\theta). \tag{3.6}$$

**Proof:**

$$E(e^{\theta S_{n+k}}) = \sum_{I \in \mathcal{I}_n} \int_I e^{\theta S_n} e^{\theta(S_{n+k} - S_n)} dt$$

where $\mathcal{I}_n$ stands for the set of q-adic intervals of order $n$. On each $I \in \mathcal{I}_n$ the function $e^{\theta S_n}$ is approximately constant so that

$$E(e^{\theta S_{n+k}}) \sim \sum_{I \in \mathcal{I}_n} \frac{1}{|I|} \int_I e^{\theta S_n} \int_I e^{\theta(S_{n+k} - S_n)}.$$

In the second integral we change variables and put $u = q^n t$ to get

$$\int_I e^{\theta(S_{n+k} - S_n)} = |I| M_k(\theta)$$

and the lemma follows by summation over $\mathcal{I}_n$. □

**Remark:** We may assume that $C$ is independent of $\theta$ if $\theta$ belongs to a bounded interval.

**Lemma 3.2.** *For any bounded interval $K \subset \mathbb{R}$ there exists $C > 0$ such that for any $\theta \in K$*

$$\int e^{\theta S_{T_N} - \log M_{T_N}(\theta)} dt \geq C \tag{3.7}$$

**Proof:** Since $\varphi$ is bounded, the variable $T_N$ is bounded. Thus there exists $k_N \in \mathbb{N}$ (depending on $t$) such that $T_N + k_N = M$ =constant. We may then write, for $I \in \mathcal{I}_n$ on which $T_N = n$ and for $k = k_N$,

$$\int_I e^{\theta S_{n+k} - \log M_{n+k}(\theta)} \sim \frac{1}{|I|} \int_I e^{\theta S_n} \int_I e^{\theta(S_{n+k} - S_n) - \log M_{n+k}(\theta)}$$

$$\sim \int_I e^{\theta S_n} \frac{M_k(\theta)}{M_{n+k}(\theta)} \sim \int_I e^{\theta S_n - \log M_n(\theta)}$$

by Lemma 3.1, and Lemma 3.2 follows by summation over all q-adic intervals on which $T_N$ is constant. □

We may now conclude the proof of (1.6). By Lemma 3.2

$$\int e^{\theta(S_{T_N} - T_N \log q) + \theta T_N \log q - \log M_{T_N}(\theta)} \geq C,$$

and (3.4) implies

$$\int e^{(\theta \log q - \frac{\log M_{T_N}(\theta)}{T_N}) T_N} \geq C e^{\theta N}.$$

Thus (2.1) and (3.5) show that

$$r(U,0) - r(U_\epsilon,0) \geq \epsilon^{-\theta}$$

provided that $\theta$ satisfies

$$\theta \log q - \frac{\log M_{T_N}(\theta)}{T_N} \leq -\log q. \tag{3.8}$$

To find such a value $\theta$, we apply the following theorem of Rohde ([4] p.191):

**Theorem 3.** *There exists $C > 0$ such that if $q \geq 3$ , $n \geq 0$ and $\theta \in [-1,0]$ then*

$$M_n(\theta) \geq Ce^{n\log(I_0(\alpha\theta))}$$

*where $I_0(x) = \frac{1}{\pi}\int_0^\pi \mathrm{ch}(x\cos t)dt$.*

It follows that every $\theta \in (-1,0)$ satisfying $\log(I_0(\alpha\theta)) = (\theta+1)\log q$ also satisfies (3.8). The existence of such $\theta$ is immediate from the intermediate value theorem. Moreover, using the computations on p.190 of [4], we find numerically, choosing $q = 11,\ \alpha = 1/B_q$, that one can take $\theta = -0.91$. □

# 4 Proof of (1.5)

We fix $\varphi \in S$ and write $r_n = 1-2^{-n}$. We recall that $U = \varphi(\mathbb{D})$ and $U_\epsilon = \varphi(D_\epsilon)$. Consider the dyadic decomposition of the disk and denote $\mathcal{W}$ the collection of all $T(Q)'s$ that meet $\partial D_\epsilon$.

**Lemma 4.1.** *There exists $C \geq 1$ such that*

$$\frac{1}{C}\sum_{\mathcal{W}} |T(Q)| \leq |\mathbb{D}\backslash D_\epsilon| \leq C\sum_{\mathcal{W}} |T(Q)|.$$

**Proof:** The first inequality follows from (2.3), and the second inequality is obvious with $C = 2$ since every point in $\mathbb{D}\backslash D_\epsilon$ is in some $Q$ such that $T(Q) \in \mathcal{W}$. □

For $n \geq 1$ let $m_n = m_n(\epsilon)$ be the number of $T(Q)'s$ in $\mathcal{W}$ with order $n$, so that

$$|\mathbb{D}\backslash D_\epsilon| \sim \sum_{n\geq 1} m_n 2^{-2n}. \tag{4.1}$$

For $z \in T(Q) \in \mathcal{W}$ we have $|\varphi'(z)|^p \sim \epsilon^p(1-|z|)^{-p}$. Writing $r_n = 1-2^{-n}$, we deduce that

$$m_n\epsilon^p 2^{np}2^{-n} \leq C\int_{|z|=r_n} |\varphi'|^p \leq C2^{n(\beta_\varphi(p)+o(1))} \tag{4.2}$$

where

$$\beta_\varphi(p) = \limsup_{r\to 1-0} \frac{\log(\int_{|z|=r} |\varphi'|^p)}{\log(\frac{1}{1-r})}.$$

More precisely, for any $\alpha > 0$ and for all $n \geq N(\alpha)$, (4.2) is true with $o(1)$ replaced by $\alpha$.

We obtain from (4.2) that $m_n \leq C\epsilon^{-p}2^{(1+\beta-p)n}$ for any $\beta > \beta_\varphi(p)$ and $\epsilon$ small enough. Together with (4.1) we obtain

$$|\mathbb{D}\backslash D_\epsilon| \leq C\epsilon^{-p}\sum_{n\geq 1} 2^{(\beta-p-1)n} \leq C\epsilon^{-p}$$

if $\beta - p - 1 < 0$. By a result of Clunie and Pommerenke ([4], p.178),

$$\beta_\varphi(p) \leq p - \frac{1}{2} + (4p^2 - p + \frac{1}{4})^{\frac{1}{2}}$$

for all $p \in \mathbb{R}$. An easy computation then shows that

$$\beta_\varphi(p) - p - 1 < 0 \quad \text{if} \quad p > \frac{1-\sqrt{33}}{8} = -0.59.. \tag{4.3}$$

This proves (1.5). $\square$

We finish by establishing the sharp exponent in Theorem 1, assuming the validity of Kraetzer's conjecture ([2]) asserting that $\beta(p) = \sup\{\beta_\varphi(p) : \varphi \in S,\ \varphi \text{ bounded}\} = p^2/4$ for all $p \in [-2, 2]$. Throughout the remainder of the paper, we will thus assume that $\beta(p) = p^2/4$.

First, using $\beta_\varphi(p) \leq p^2/4$ instead of the Clunie-Pommerenke estimate in (4.3), we get $\beta_\varphi(p) - p - 1 \leq \frac{p^2}{4} - p - 1$ and this last quantity is negative for $-2(\sqrt{2}-1) = -0.828.. < p < 0$. It follows that for any simply connected domain $U$ with $r(U,0) = 1$ and for all $\alpha > 0$, $r(U,0) - r(U_\epsilon, 0) \leq C_\alpha \epsilon^{2(\sqrt{2}-1)-\alpha}$.

Conversely, we will show that for any $\alpha > 0$ there is a domain $U$ such that $r(U,0) - r(U_\epsilon,0) \geq C_\alpha\epsilon^{2(\sqrt{2}-1)+\alpha}$. Set $p = -2(\sqrt{2}-1)$. It is known (fractal approximation, see [3]) that for any $\delta > 0$ there exists a conformal map $\varphi$ with $\varphi'(0) = 1$ such that

$$\int_{|z|=r_n} |\varphi'|^p \geq C_\delta 2^{n(\beta(p)-\delta)} \quad \text{for all } n \geq 1.$$

Using Koebe we then write

$$\int_{|z|=r_n} |\varphi'|^p \sim \sum_k 2^{-n}|\varphi'(z_k)|^p$$

where the $z_k' s$ are the dyadic points of order $n$ on $\{|z| = r_n\}$. Fixing a small positive number $\gamma > 0$, we partition the set of $z_k' s$ into the sets $\mathcal{B}, \mathcal{A}_j$, where

$$\mathcal{B} = \{z_k : |\varphi'(z_k)|^{-1} \leq 1\},\ \mathcal{A}_j = \{|\varphi'|^{-1} \in [2^{nj\gamma}, 2^{n(j+1)\gamma})\},\ 0 \leq j \leq \frac{2}{\gamma}.$$

We may write $\sum_k 2^{-n}|\varphi'(z_k)|^p = \sum_{k\in\mathcal{B}} + \sum_j \sum_{k\in\mathcal{A}_j}$. Since $\sum_{\mathcal{B}} \leq 1$ there must exist $j$ such that

$$\sum_{k\in\mathcal{A}_j} 2^{-n}|\varphi'(z_k)|^p \geq C\gamma \int_{|z|=r_n} |\varphi'|^p \geq C\gamma 2^{n(\beta(p)-\delta)}.$$

We fix such a $j = j_n$ and denote by $m_n$ the number of elements of $\mathcal{A}_j$. Setting $a = a_n := j\gamma$ we then have

$$m_n 2^{-n} 2^{-n(j+1)\gamma p} \geq C\gamma 2^{n(\beta(p)-\delta)}.$$

Consequently

$$m_n \geq 2^{-n(-\beta(p)-1-ap+2\delta)}$$

if $\gamma$ is chosen small enough and $n$ is large. On the other hand, for $k \in \mathrm{A}_j$,

$$2^{-n}|\varphi'(z_k)| \leq 2^{-n}2^{-nj\epsilon} = 2^{-n(1+a)} =: \epsilon_n.$$

Therefore

$$|\mathbb{D}\backslash D_{\epsilon_n}| \geq C m_n 2^{-2n} = C\epsilon_n^\lambda$$

where

$$\lambda = \lambda_n = \frac{1 - pa - \beta(p) + 2\delta}{1+a}.$$

Since $\frac{1-pa-\beta(p)}{1+a} \equiv -p$, the claim follows by choosing $\delta$ arbitrarily small.

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