Dimension distortion of hyperbolically convex maps

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Abstract

In this note, we provide an answer to a question of D. Mejia and Chr. Pommerenke, by constructing a hyperbolically convex subdomain G of the unit disc $\mathbb{D}$ so that the conformal map from $\mathbb{D}$ to G maps a set of dimension 0 on $\partial\mathbb{D}$ to a set of dimension 1.

1 Introduction and statement of the result

A subdomain G of the unit disc $\mathbb{D}$ in $\mathbb{C}$ is *hyperbolically convex* if it is convex with respect to the hyperbolic metric of $\mathbb{D}$. These domains are just intersections of hyperbolic halfplanes. Conformal maps from $\mathbb{D}$ onto hyperbolically convex domains have been systematically studied beginning with Ma and Minda ([2] and [3]), see the papers [4], [5] and [6] by Mejia and Pommerenke for further references. Hyperbolically convex domains appear naturally in various situations such as the Hayman-Wu problem, where Brown-Flynns theorem (the boundary of hyperbolically convex domains has length $< \pi^2$) has been used by Fernández, Heinonen and Martio [1] to obtain the bound $4\pi^2$ for the Hayman-Wu constant (see [8] for the best bound to date and for references).

In [4], Mejia and Pommerenke proved that, for conformal maps f of $\mathbb{D}$ onto hyperbolically convex domains,

$$\dim f(A) < 1$$

if $A \subset \mathbb{T}$ has $\dim A < 1$ and if $f(A)$ is uniformly perfect. They asked (Conjecture 3 in [5]) if the assumption of uniform perfectness can be omitted. We will answer this question to the negative by proving

Theorem 1. *There is a hyperbolically convex domain $G \subset \mathbb{D}$ and a set $A \subset \mathbb{T}$ with $\dim A = 0$ and $\dim f(A) = 1$.*

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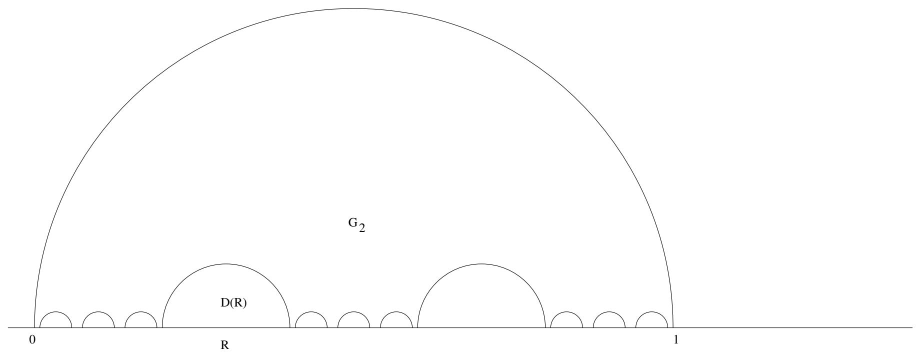

Figure 1.1: The second approximation to the hyperbolically convex domain G.

Our domain is of Schottky type (meaning that the complement of $\overline{G}$ is a union of *disjoint* hyperbolic halfplanes; see [4] and Theorem 2 therein). The inductive construction of the domain G as a descreasing sequence of hyperbolically convex domains G_n is slightly easier to describe in the upper halfplane $\mathbb{H}$: Fix a sequence $\varepsilon_1, \varepsilon_2, ...$ of small positive numbers and positive integers $n_1, n_2, ...$ (to be determined later). Next, construct a Cantor-like set $S \subset [0,1] \subset \mathbb{R}$ in the following way: Set $S_0 = [0,1]$. Suppose that S_k has already been constructed and consists of $N_k = n_1 n_2 \cdots n_k$ intervals $I_j^{(k)}$ of size

$$s_k = (\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k)/(n_1 n_2 \cdots n_k).$$

Form S_{k+1} from S_k by replacing each of the $I_j^{(k)}$ by n_{k+1} equally spaced subintervals $I_\ell^{(k+1)}$ of size $s_{k+1} = (\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{k+1})/(n_1 n_2 \cdots n_{k+1})$. Set

$$S = \bigcap_k \bigcup_j I_j^{(k)}.$$

For instance, the middle-third Cantor set is the special case $n_k = 2$ and $\varepsilon_k = 2/3$. Denote the "removed" invervals by $R_\ell^{(k+1)}$, $1 \le \ell \le (n_{k+1} - 1) n_1 \cdots n_k$. For an interval $I \subset [0,1]$, let $D(I)$ be the open halfdisc in $\mathbb{H}$ with diameter I. Set

$$G_k = D([0,1]) \setminus \bigcup_{1 \le m \le k} \bigcup_j \overline{D(R_j^{(m)})}$$

and

$$G = \bigcap G_k,$$

see Figure 1.1.

Fix a point $z_0 \in G$. Let g_k respectively g denote the conformal map of G_k respectively G onto $\mathbb{D}$ normalized by $g_k(z_0) = 0$ and $g_k'(z_0) > 0$. Because ∂G is a simple curve, g extends continuously to $\overline{G}$. We will show that appropriate choice of the parameters ε_k and n_k yields a set S of dimension 1 such that $g(S)$ has dimension 0, proving the theorem (with $f = g^{-1}$ and $A = g(S)$). More precisely, we will prove

Theorem 2. *If*

$$\log n_k = o(\frac{1}{\sqrt{\varepsilon_k}}) \quad as \quad k \to \infty,$$

then

$$\dim g(S) = 0.$$

On the other hand, if

$$\log \frac{1}{\varepsilon_k} = o(\log n_k) \quad as \quad k \to \infty,$$

then

$$\dim S = 1.$$

Theorem 1 follows immediately by taking any pair of sequences ε_k, n_k that satisfies both conditions, for instance $\varepsilon_k = 1/k^2$ and $n_k = \exp(\sqrt{k})$.

It was realized in [4] that domains of Schottky type are the "critical domains" for the validity of the dimension distortion property: Mejia and Pommerenke proved (Theorem 2 of [4]) that $\dim A < 1$ would imply $\dim f(A) < 1$ for *all* hyperbolically convex f if $\dim A < 1$ implies $\dim f(A) < 1$ for *those* hyperbolically convex f that are *of Schottky type.* Notice that our domain G is of Schottky type, so that the assumption of their Theorem 2 does not hold.

The phenomenon responsible for the strong dimension-distortion is the following: If ε_k is very small and I is one of the $I_j^{(k)}$, then near I the domain G_k looks like a half-strip, whose sides are the two geodesics ending in the endpoints of I. Therefore the harmonic measure of I seen from z_0 is exponentially smaller than the size s_k of I. For our set S to have the properties $\dim S = 1$ and $\dim g(S) < 1$ we will need $\varepsilon_k \to 0$. Thus S cannot be uniformly perfect, in accordance with Theorem 3 of [4].

2 Proofs

The letter C denotes various constants whose value may change even within a line. Recall the definition of S from Section 1 and assume that $\varepsilon_j < \frac{1}{2}$ for all j. Denote $\hat{I}_j^{(k)}$ the (smaller) arc of ∂G that has the same endpoints as $I_j^{(k)}$. Fix $z_0 \in G$.

Lemma 2.1. *For all j and k we have*

$$\omega(z_0, \hat{I}_j^{(k)}) \leq K_1 \exp{(-K_2 \sum_{\ell=1}^{k} \frac{1}{\sqrt{\varepsilon_\ell}})},$$

where K_2 is a universal constant and K_1 depends on z_0 and G only, but not on j or k.

Proof: Fix an interval $I = I_{j_0}^{(k_0)}$. Denote r_k the length of the intervals $R_j^{(k)}$. Then $(n_{k+1}-1)r_{k+1} + n_{k+1}s_{k+1} = s_k$ so that we have

$$r_{k+1} = \frac{1-\varepsilon_{k+1}}{n_{k+1}-1}s_k > \frac{\varepsilon_{k+1}}{n_{k+1}}s_k = s_{k+1}.$$

It follows that

$$s_{k+1} < h_{k+1} := \sqrt{r_{k+1}s_{k+1}} < r_{k+1} < s_k. \tag{2.1}$$

Let Γ denote the family of all curves γ in G that join a small (fixed) circle centered at z_0 with $\hat{I}_j^{(k)}$. By conformal invariance of the modulus $M(\Gamma)$ and by Pflugers theorem (see for instance [7], Chapter 9) we have

$$\omega(z_0, \hat{I}_j^{(k)}) \leq C\exp(-\frac{\pi}{M(\Gamma)}),$$

where C depends on z_0 and the circle but not on j or k. Consider the strips $S_k = \{x+iy : s_k < y < h_k\}$. It follows from (2.1) that they are pairwise disjoint. For $1 \leq k \leq k_0$, denote Q_k the component of $G \cap S_k$ that separates I from z_0. (We will assume without much loss of generality that $\Im z_0 > h_1$). Each Q_k is, or is contained in, a topological quadrilateral bounded by two horizontal line segments (top and bottom) and two circular arcs (sides) of circles of diameter at least r_k. The length ℓ of the top is maximal if both sides come from circles of diameter r_k. Because both circles are separated by an interval $I_j^{(k)}$ of size s_k, it follows that

$$\ell \leq s_k + 2(r_k/2 - \sqrt{(r_k/2)^2 - h_k^2})$$

and it thus follows from (2.1) using $\sqrt{1-x} \geq 1-x$ that

$$\ell < 5s_k.$$

Denote Q'_k the smallest rectangle containing Q_k: It has horizontal sides of length ℓ and vertical sides of length $h_k - s_k$. Every curve $\gamma \in \Gamma$ has to pass through each of the Q'_k $(1 \leq k \leq k_0)$ and therefore (e.g. [7], Proposition 9.3)

$$\frac{1}{M(\Gamma)} \geq \sum_{k=1}^{k_0} \frac{1}{M(Q'_k)} = \sum_{k=1}^{k_0} \frac{h_k - s_k}{\ell_k} \geq \sum_{k=1}^{k_0} \frac{h_k}{5s_k} \geq \frac{1}{5\sqrt{2}} \sum_{k=1}^{k_0} \frac{1}{\sqrt{\varepsilon_k}},$$

where the last inequality follows using $r_k/s_k = (n_k/\varepsilon_k)(1-\varepsilon_k)/(n_k-1)$. The Lemma follows from Pflugers theorem, with $K_2 = \pi/(5\sqrt{2})$. □

Proof of Theorem 2: For every k, we have

$$g(S) \subset \bigcup_j g(\hat{I}_j^{(k)}).$$

By Lemma 2.1, the arc $g(\hat{I}_j^{(k)})$ has length

$$\ell_{j,k} \leq 2\pi K_1 \exp\left(-K_2 \sum_{\ell=1}^{k} \frac{1}{\sqrt{\varepsilon_\ell}}\right).$$

For every $\alpha > 0$,

$$\sum_j \ell_{j,k}^{\alpha} \leq C n_1 n_2 \cdots n_k \exp\left(-K_2 \alpha \sum_{\ell=1}^{k} \frac{1}{\sqrt{\varepsilon_\ell}}\right) = C \prod_{\ell=1}^{k} \left(n_\ell \exp\left(-K_2 \alpha \frac{1}{\sqrt{\varepsilon_\ell}}\right)\right)$$

and the assumption $\log n_j = o(\frac{1}{\sqrt{\varepsilon_j}})$ implies that the factors go to zero as $j \to \infty$. Thus $\dim g(S) \leq \alpha$.

To prove that $\dim S = 1$ if $\log(1/\varepsilon_j) = o(\log n_j)$, fix $\alpha < 1$ and consider the measure μ on S defined by

$$\mu(I_j^{(k)}) = \frac{1}{n_1 n_2 \cdots n_k}$$

for all j and k. The theorem follows at once from Frostman's Lemma if we can prove that, for every interval I, we have

$$\mu(I) \leq C_\alpha |I|^\alpha. \tag{2.2}$$

To this end, let k be such that $s_{k+1} \leq |I| < s_k$. Because $r_k > s_k$, there is only one index j such that $I_j^{(k)} \cap I \neq \emptyset$. Let $t = |I|/s_k < 1$. Then I can intersect at most $tn_{k+1} + 2$ of the n_{k+1} subintervals $I_\ell^{(k+1)}$ of $I_j^{(k)}$. Therefore

$$\mu(I) \leq (tn_{k+1} + 2) \frac{1}{n_1 n_2 \cdots n_{k+1}}. \tag{2.3}$$

By assumption,

$$\log \frac{1}{\varepsilon_p} \leq \frac{1-\alpha}{\alpha} \log n_p$$

for p large enough, so that

$$\frac{1}{n_1 n_2 \cdots n_k} \leq C \left(\frac{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_k}{n_1 n_2 \cdots n_k}\right)^\alpha = C s_k^\alpha.$$

Now (2.3) implies

$$\mu(I) \leq C(t s_k^\alpha + 2 s_{k+1}^\alpha) \leq C(t^{1-\alpha} |I|^\alpha + 2|I|^\alpha)$$

and (2.2) is proved. □

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