Conformal laminations and Gehring trees

Steffen Rohde*

Dedicated to the memory of Frederick W. Gehring

Preliminary version

Abstract

We characterize the laminations that arise from conformal mappings onto those planar John domains whose complements have empty interior. This generalizes the well-known characterization of quasicircles via conformal welding. Our construction of a John domain from a given lamination introduces a canonical realization of finite laminations, and generalizes to some Hölder domains.

1 Introduction and Results

This paper deals with the question of characterizing those equivalence relations on the circle that arise from conformal maps f of the unit disc onto simply connected plane domains, where x and y are declared equivalent if $f(x) = f(y)$ (assuming they exist as radial limits, say). There are several motivations for this question, discussed at the end of this section. While a complete characterization seems to be out of reach, this paper introduces tools that allow for a characterization of a large and natural class of such laminations. Our approach is heavily influenced by Chris Bishop's approach to welding.

We define a *Gehring tree* $T \subset \mathbb{C}$ to be a dendrite (connected, locally connected, compact without non-trivial simple loops) whose complement is a John-domain, see Section 2.2 for details, equivalent definitions and properties. Prominent examples are the Julia sets of semihyperbolic polynomials [9] such as $z^2 + i$. Every smoothly embedded finite tree with non-zero angles at the vertices, and every quasiconformal arc is a Gehring tree as well.

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By Caratheodory's theorem, every compact, connected and locally connected subset $T \subset \mathbb{C}$ induces a *lamination* $\mathcal{L}$ of the disc $\mathbb{D}$ via the hydrodynamically normalized conformal map $f_T : \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \overline{\mathbb{C}} \setminus T$ by setting

$$\mathcal{L} = \{(z, w) \in \mathbb{T} \times \mathbb{T} : f_T(z) = f_T(w)\}.$$

The term lamination will always refer to a closed and flat equivalence relation on $\mathbb{T}$. Here it closed means that the support of $\mathcal{L}$, namely the set of endpoints z, w of the leaves $(z, w) \in \mathcal{L}$, is closed as a subset of $\mathbb{T}$, and *flat* means that leaves do not intersect until their endpoints are equivalent. It is *maximal* if there is no $(z, w) \notin \mathcal{L}$ such that $\mathcal{L} \cup \{(z, w)\}$ is a lamination. and *non-degenerate* if all equivalence classes are totally disconnected. The main result of this paper is the following characterization of Gehring trees in terms of their laminations, see Figures 1.1 and 3.2, and see Section 2.1 for notation. It essentially proves an unpublished conjecture of Chris Bishop and Peter Jones.

Theorem 1.1. *A maximal, non-degenerate lamination $\mathcal{L}$ is the conformal lamination of a Gehring tree if and only if it satisfies the following condition: There are constants C and N such that for every point $x \in \mathbb{T}$ and every scale $0 < r < 1$ there are pairs of disjoint adjacent intervals I_j, I'_j, $j = 1, ..., n$ (possibly $n = 1$) with $n \leq N$, uniformly perfect subsets $A_j \subset I_j, A'_j \subset I'_j$, and decreasing quasisymmetric homeomorphisms $\phi_j : A_j \to A'_{j+1}$ such that $I_1 \cap I'_1 = \{x\}$, $|I_1| \asymp_C r$, $|I_j| \asymp_C |I'_j| \asymp_C \operatorname{diam} A_j \asymp_C \operatorname{diam} A'_j$ for all j ($A'_{n+1} := A'_1$), and such that $(t, \phi_j(t)) \in \mathcal{L}$ for all $t \in A_j$. This is quantitative in the sense that all constants (C,N, quasisymmetry, uniform perfectness) are bounded by the constant of $\mathcal{L}$ and vice-versa.*

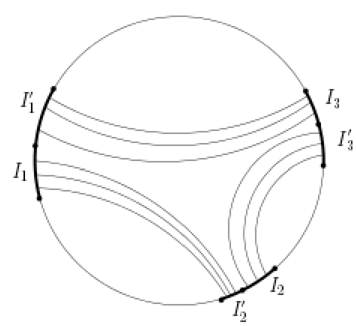

Figure 1.1: The situation of Theorem 1.1 near a triple point, see also Figure 3.2.

It follows from P. Jones' removability theorem [19] that the dendrite of Theorem 1.1 is unique up to a linear map: Indeed, given any two realizations f and g of the same lamination $\mathcal{L}$, the conformal map $g \circ f^{-1}$ of the complement of the first dendrite has a homeomorphic extension to the plane and hence is linear. By the Jones-Smirnov removability theorem [20], this argument generalizes to dendrites whose complement is a Hölder domain.

The strategy of the proof can be summarized as follows. John domains can be viewed as one-sided quasidiscs (see Section 2.2) and have good localization properties, see [19] for important examples. For the necessity of the condition, we give a geometric construction (Proposition 3.5) of a localization that provides a decomposition of an annulus into a chain of boundedly many topological squares as in Figure 3.2. The boundaries of two consecutive squares are connected through a subset of the tree. Typically, the harmonic measures of different "sides" of the tree are typically mutually singular, in other words the set of biaccessible points is of harmonic measure zero. This is dealt with by proving the existence of a large set (in the sense of potential theory) of biaccessible points. Another difficulty is that the size alone of the set of biaccessible points provides no information: Indeed, in the classical welding problem (trees without branching) every point is biaccessible. What is needed in addition is control over the *quality* of the welding. We show that there is a large subset of the set of biaccessible points on which the welding is quasisymmetric. This relies on a result of Väisälä that conformal maps between John domains are quasisymmetric in the *internal* metric, combined with a construction of a subset of the tree where the euclidean and the internal metric are comparable.

The proof of the sufficiency relies on the following key observation (Proposition 3.2): In order to control the conformal modulus of the welding of two squares along an edge, it is enough to require quasisymmetry of the glueing on a sufficiently large subset (heuristically, this can be viewed as a generalization of the fact that glueing two Loewner spaces isometrically along their boundaries yields a Loewner space). We will not use Proposition 3.2 explicitely, but its proof illustrates several features of our proof of the theorem and can be viewed as a toy case.

Next, we construct an approximation $\mathcal{L}_n$ of finite laminations to the given lamination $\mathcal{L}$. Another key idea is to realize any finite lamination by a canonical domain that we call a *balloon animal*, Proposition 4.1 and Figure 4.1: It is characterized by the property that harmonic measure at ∞ and the harmonic measures of the bounded components G_i are linearly related,

$$\frac{d\omega_\infty}{d\omega_i} \equiv \text{const} \quad \text{on} \quad \partial G_i.$$

This provides the tool for estimating the conformal modulus of the annuli obtained from glueing a chain of squares along (subsets of) their edges: Roughly speaking, loops surrounding the annulus described above correspond to loops that pass through the balloons in the discrete approximation, and the defining property of the ballon animal allows to pass information from the complement of a ballon (measured by ω_∞) to the interior of the balloon (measured by ω_i) without distortion. It is then fairly standard to translate such modulus estimates into analytic control.

While John domains are "tame" at every scale, the larger class of Hölder domains has similar regularity properties but allows for more pathological behaviour (for instance bottlenecks). For instance, domains bounded by SLE-like paths are Hölder domains but not John domains, almost surely. Similarly, the dendrites arising as Julia sets of quadratic polynomials for parameters c in the boundary of the Mandelbrot set M are Hölder domains but not John domains, almost surely with respect to harmonic measure of the complement of M. For this reason it could be interesting to note that the proof of the sufficiency of the condition easily generalizes to Hölder domains. Say that a lamination $\mathcal{L}$ has $(C,N)-$good glueing near x at scale r if the condition of Theorem 1.1 is satisfied.

Corollary 1.2. *If there are constants C,N and μ such that a maximal, non-degenerate lamination $\mathcal{L}$ has $(C,N)-$good glueings with frequency μ,*

$$\frac{\#\{k : 1 \leq k \leq m, \mathcal{L} \text{ has } (C,N) - \text{good glueing near } x \text{ at scale } 2^{-k}\}}{m} \geq \mu \quad \text{for all } x \text{ and } m,$$

then $\mathcal{L}$ is the conformal lamination of a dendrite whose complement is a Hölder domain, quantitatively.

It would be interesting to find a characterization of Hölder domains in terms of their lamination. Another corollary is as follows:

Corollary 1.3. *If $\mathcal{L}$ is a maximal, non-degenerate lamination and if $\mathcal{L}' \subset \mathcal{L}$, then $\mathcal{L}'$ is conformal and there is a conformal map onto a John domain realizing $\mathcal{L}'$.*

We will now discuss some motivations and open questions.

Conformal and quasiconformal mapping:

Conformal welding of Jordan curves plays an important role in geometric function theory. The problem of conformally realizing a given lamination is a natural generalization. Without any regularity assumption on the lamination, an understanding seems to be out of reach, even in the simpler classical case of Jordan curves. On the other hand, quasisymmetric homeomorphisms of the circle form an important, natural, large and well-studied class that admit welding. Theorem 1.1 could be viewed as identifying the most natural generalization to laminations, at least from the quasiconformal viewpoint.

Don Marshalls zipper algorithm [25] (see also [26]) provides numerical solutions to conformal welding problems and is remarkably accurate, particularly in the setting of quasisymmetric weldings. In the forthcoming [27], we describe a closely related zipper algorithm that allows to numerically approximate conformal realizations of laminations, and give applications to Shabat polynomials. The numerical accuracy of Don Marshall's implementation of this algorithm is most remarkable. It seems reasonable to believe, and is supported by the numerical computations for trees with thousands of edges discussed in [27], that the algorithm converges at least for Gehring trees.

Complex dynamics:

Beginning with Thurston [36] and Douady-Hubbard [11], conformal laminations of Julia sets of quadratic polynomials have been studied and used to give combinatorial models of both Julia sets and the Mandelbrot set. The question which laminations allow for a conformal realization has been raised in numerous places and has been identified as difficult, see for instance the comments in [10] and [35]. Some results about existence in a setting somewhat dual to ours, namely that of small support of $\mathcal{L}$ (logarithmic capacity zero of the set of endpoints) can be found in the Ph.D. theses [23] and [15]. In the setting of quadratic Julia sets, it has been proved that the set of bi-accessible points is always of harmonic measure zero [38],[35] and even of dimension less then one [28], except for the Tchebyscheff polynomial $z^2 - 2$. In a different direction, Carleson, Jones and Yoccoz [9] have shown that the domain of attraction to ∞ is a John-domain if and only if the polynomial is semi-hyperbolic, namely the critical point 0 is non-recurrent (and there is no parabolic point). The most prominent examples are the post-critically finite polynomials such as $z^2 + i$, where the critical point is pre-periodic. The conditions of Theorem 1.1 can be verified directly in these important cases: With Peter Lin [24] we have given a combinatorial description of semi-hyperbolicity and prove

Theorem 1.4. *The lamination of a combinatorially semi-hyperbolic quadratic polynomial satisfies the condition of Theorem 1.1, hence the Julia set is a Gehring tree.*

In fact, in this case the uniformly perfect sets can be chosen as linear Cantor sets, and the quasisymmetric homeomorphisms are linear maps. As a corollary, we obtain a new proof of the Carleson-Jones-Yoccoz theorem (in the quadratic setting).

In [17], Hubbard and Schleicher gave a proof of convergence of the spider-algorithm in the *periodic* case of critically finite quadratics. In [24] we relate the spider-algorithm to the balloon animals and obtain as another corollary

Corollary 1.5. *The Hubbard-Schleicher spider-algorithm converges for postcritically finite quadratic polynomials.*

Probability theory

A main motivation comes from potential applications to questions related to random maps originating in probability theory. Every finite (ordered) tree has an embedding in the plane, unique up to linear maps, as a *balanced tree* in Bishop's terminology [8]. Balanced trees are related to Shabat polynomials and are special cases of Grothendieck dessin d'enfant: There is a polynomial whose only finite critical values are 0 and 1 such that the preimage $p^{-1}([0,1])$ is an embedding of the combinatorial tree. They are of interest to different groups of mathematicians, partly because of the faithful action of the absolute Galois group of the rationals on dessins, and partly because of the challenges posed by their computations. We are interested in large random trees. The term *balanced* refers to the fact that each edge has the same harmonic measure at infinity, and the harmonic measures on both sides of each edge are identical. Consequently, balanced trees realize very special laminations, namely those that are associated with non-crossing partitions of the $2n-$gon, see [8],[6] and [27].

Recent groundbreaking progress on the mathematical side of quantum gravity, particularly the convergence of scaled large random planar maps to the Brownian map as the number of faces tends to infinity (Le Gall [22], Miermont [29]) and the work of Miller and Sheffield on the relation of the Brownian map to Liouville Quantum Gravity (see [30] and the references therein), suggest that conformal realizations of large or infinite random trees are of interest, and that a conformal structure of the Brownian map can be thought of as the conformal mating of two independent random trees. It has been asked by Le Gall (oral communication) if the Brownian lamination is conformal, see [27] for simulations. We believe that even more is true:

Conjecture 1.6. *The Brownian lamination $\mathcal{L}$ is conformal, and the associated conformal map $f_{\mathcal{L}}$ is Hölder continuous with probability one. It is the distributional limit of the uniform balanced trees on n edges as $n \to \infty$.*

We hope that Theorem 1.1 is a step towards a proof of this conjecture, and believe that the methods developed in this paper will be useful in less regular settings.

1.1 Acknowledgement

This paper evolved over several years and could be viewed as part of a larger ongoing project with Don Marshall, whom I would like to thank for countless inspiring discussions. In an early stage, a collaboration with James Gill and Don Marshall on questions related to this paper led to a proof of the sunny Lemma 3.9 in the case $N = 1$. I would like to thank them for this collaboration.

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2 Preliminaries

In this section we collect definitions and facts about John domains, quasisymmetric maps, logarithmic capacity, conformal modulus and uniformly perfect sets, as can be found for instance in the monographs [1], [2], [4], [12], [16], [33]. We also prove a few technical results that will be needed later on, particularly Proposition 2.12. The expert may skip this section and return to these results when needed.

2.1 Notation

Throughout this paper we will use the following notation:

$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended complex plane, $\mathbb{D}$ is the (open) unit disc, $\Delta = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, $\mathbb{T} = \partial\mathbb{D}$ the unit circle. $D_r(z) = D(z,r)$ is the disc of radius r centered at z, $C_r(z) = \partial D_r(z)$, and $C_r = \partial D_r(0)$.

We denote line segments by $[a,b]$, arcs (intervals) on $\mathbb{T}$ by (a,b), and geodesics in hyperbolic domains by $< a, b >$, and the length of a line segment or arc by $|I|$.

We write $a \asymp b$ and sometimes more explicitely $a \asymp_C b$ to designate the existence of a constant C such that $1/C \leq a/b \leq C$. We sometimes say that a statement holds *quantitatively* if the associated parameters (domain constants etc.) only depend on the parameters associated with the data.

2.2 John domains, quasisymmetric maps and Gehring trees

A connected open subset D of the Riemann sphere is a *John-domain* if there is a point $z_0 \in D$ (the John-center) and a constant C (the John-constant) such that for every $z \in D$ there is a curve $\gamma \subset D$ from z_0 to z such that

$$\mathrm{dist}(\gamma(t), z) \leq C \mathrm{dist}(\gamma(t), \partial D)$$

for all t. If $\infty \in D$, then $z_0 = \infty$. An equivalent definition ([33]) is that

$$\mathrm{diam}\, D(\sigma) \leq C' \,\mathrm{diam}\, \sigma \tag{2.1}$$

for every crosscut σ of D, where $D(\sigma)$ denotes the component of $D \setminus \sigma$ that does not contain z_0. Moreover, it is enough to consider crosscuts that are line segments.

John domains were introduced in [18] and are ubiquitous in analysis. Simply connected planar John domains can be viewed as one-sided quasidiscs. Indeed, a Jordan curve is a quasicircle if and only if both complementary components are John domains. Our main reference is the exposition [31] of Näkki and Väisälä. Important work related to John domains can be found in [3],[19],[9],[31],[37] and a large number of references in these works. A *planar dendrite* is a compact, connected, locally connected subset T of the plane $\mathbb{C}$ with trivial fundamental group.

Definition 2.1. A Gehring tree is a planar dendrite such that the complement is a John-domain.

Gehring trees are easily described as planar dendrites built from quasiconformal arcs, see [2], [4] and [16] for basic definitions and an introduction to quasiconformal maps. A *$K-$quasiarc* is the image of a straight line segment under a $K-$quasiconformal homeomorphism of the plane. For every arc γ and all $x, y \in \gamma$, denote $\gamma(x, y)$ the subarc with endpoints x, y. Quasiarcs are characterized by Ahlfors'-condition

$$\operatorname{diam} \gamma(x,y) \leq K|x-y| \quad \text{for all} \quad x, y, \tag{2.2}$$

see [13] for a wealth of properties and characterizations of quasiconformal arcs and discs.

Proposition 2.2. *A dendrite T is a Gehring-tree if and only if there is K such that every subarc $\alpha \subset T$ is a $K-$quasiarc.*

This follows from the well-know fact [31] that the complement of a John disk is of bounded turning. We give a simple direct proof:

Proof. If $\alpha \subset T$ is a subarc and $x, y \in \alpha$, then $[x, y] \backslash T$ is a collection of intervals $\sigma_j = [x_j, y_j]$ that are crosscuts of $D = \mathbb{C} \setminus T$. It is easy to see that

$$\alpha \subset [x,y] \cup \bigcup_j \overline{D(\sigma_j)}$$

so that α satisfies the Ahlfors condition by (2.1).

Conversely, if T is a tree consisting of $K-$quasiarcs, and if $[x, y]$ is a crosscut of $D = \mathbb{C} \setminus T$, then the outer boundary of $D([x, y])$ is $[x, y] \cup T(x, y)$, and (2.1) follows from the quasiarc property of $T(x, y)$. □

The notion of *quasisymmetry* is a generalization of quasiconformality to the setting of metric spaces, see [16]. An embedding $f : X \to Y$ of metric spaces (X, d_X) and (Y, d_Y) is quasisymmetric if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that $d_Y(f(x), f(z)) \leq \eta(t) d_Y(f(y), f(z))$ whenever $d_X(x, z) \leq t d_X(y, z)$.

If $D \subsetneq \hat{\mathbb{C}}$ is open and connected, the *internal metric* is defined as

$$\delta_D(x, y) = \inf_\gamma \operatorname{diam}(\gamma),$$

where the infimum is over all curves $\gamma \subset D$ with endpoints x and y. If ∂D is locally connected, or equivalently if a conformal map f from the disc onto D has a continuous extension to $\overline{D}$, then the completion of (D, δ_D) coincides with the completion of D via the prime end boundary, and f extends to a homeomorphism between $\overline{\mathbb{D}}$ and this completion. For instance, if D is a slit disc, then both sides of the slit give rise to different points in the closure of (D, δ_D). John domains are characterized by the quasisymmetry of this extension:

Theorem 2.3 ([31], Section 7)**.** *A conformal disc D with $\infty \in D$ is a $c-$John domain if and only if the conformal map $f : \Delta \to D$ that fixes ∞ is quasisymmetric in the internal metric. Here η depends only on c and vice versa.*

John domains are intimately related to the doubling property for harmonic measure. We will use the following characterization. The proof in ([33], Theorem 5.2) for bounded domains can easily be modified to cover our situation.

Theorem 2.4. *Let $f : \Delta \to G$ be a conformal map fixing ∞. Then G is a John domain if and only if there is a constant $\beta > 0$ such that*

$$\operatorname{diam} f(A) \le \frac{1}{2} \operatorname{diam} f(I)$$

whenever $A \subset I \subset \mathbb{T}$ are arcs of length $|A| \le \beta |I|$.

Finally, we will need the following result of P. Jones, [19],[31].

Lemma 2.5. *If D is a John domain, $f : \mathbb{D} \to D$ a conformal map sending 0 to the John center, and if $D' \subset \mathbb{D}$ is a John domain, then $f(D')$ is a John domain, quantitatively.*

2.3 Logarithmic capacity, conformal modulus and uniformly perfect sets

Throughout the remainder of this section, A will be a compact subset of $\mathbb{C}$. If A posesses a Green's function $g_A(z)$, its *logarithmic capacity* $\operatorname{cap} A$ can be defined by the expansion

$$g_A(z) = \int_A \log|z-w| d\omega_\infty(w) - \log(\operatorname{cap} A) = \log|z| - \log(\operatorname{cap} A) + O(\frac{1}{|z|}),$$

where ω_∞ is the harmonic measure of $\mathbb{C} \setminus A$ at ∞. Important examples are $\operatorname{cap} B(x,r) = r$ and $\operatorname{cap}[a,b] = |b-a|/4$. The capacity of non-compact sets is defined as the supremum of the capacities of compact subsets.

Capacity and harmonic measure are quantitatively related as follows: If $A \subset D_{1/2}(0)$ and if $|z| = 1$, then

$$\omega_z(A, D_2(0) \setminus A) \asymp 1/\log \frac{1}{\operatorname{cap} A}, \tag{2.3}$$

see Theorem 9.1 in [12].

We will also need the *subadditivity* property of capacity: If $E = \cup E_n$ is of diameter ≤ 1, then

$$1/\log\frac{1}{\operatorname{cap} E} \leq \sum_n 1/\log\frac{1}{\operatorname{cap} E_n}.$$

The *conformal modulus* $M(\Gamma)$ of a family of curves Γ is an important conformal invariant, see[1],[12],[33]. It is defined as

$$M(\Gamma) = \inf_\rho \rho^2 dxdy$$

where the infimum is over all *admissable metrics* ρ, namely all Borel measurable functions ρ with the property that $\int_\gamma \rho|dz| \geq 1$ for all $\gamma \in \Gamma$.

The proof of Theorem 1.1 relies on estimates of the modulus of continuity of some conformal maps. We will employ a standard technique to obtain such estimates. It is based on the following relation between the conformal modulus of an annulus and its euclidean dimensions. Consider a topological annulus $A \subset \mathbb{C}$ with boundary components A_1, A_2 and set $r(A) = \min(\operatorname{diam} A_1, \operatorname{diam} A_2)$, $R(A) = \operatorname{dist}(A_1, A_2)$. The conformal modulus $M(A)$ is defined as the modulus of the family of all closed curves $\gamma \subset A$ that separate A_1 and A_2. For example, $M(A(x,r,R)) = \log(R/r)/2\pi$.

Lemma 2.6. *There is a constant C such that*

$$|M(A) - \frac{1}{2\pi}\log(1 + \frac{R(A)}{r(A)})| \leq C.$$

See for instance ([34], Lemma 2.1) for a discussion and references. We will use it in combination with the subadditivity property of the modulus:

Lemma 2.7. *If A_j are disjoint annuli that separate the boundary components of an annulus A, then $M(A) \geq \sum_j M(A_j)$.*

We will also use Pfluger's theorem which quantifies a close connection between capacity and modulus:

Theorem 2.8. *If $E \subset \partial\mathbb{D}$ is a Borel set and if Γ_E is the set of all curves $\gamma \subset \mathbb{D}$ joining the circle C_r to the set E, then*

$$\operatorname{cap} E \asymp e^{-\pi/M(\Gamma_E)}$$

with constants only depending on $0 < r < 1$.

Specifically, we will use the following variant whose proof we leave as an exercise: If $D = [0, X] \times [0, 1]$ is a rectangle, if $E \subset \{X\} \times [0, 1]$ is Borel, and if Γ_E is the family of curves that join $\{0\} \times [0, 1]$ to E in D, then

$$\operatorname{cap} E \leq C(X) e^{-\pi/M(\Gamma_E)}. \tag{2.4}$$

The compact set A is called *uniformly perfect* if there is a constant $c > 0$ such that no annulus $A(x, cr, r)$ with $r < \operatorname{diam} A$ separates A : If $A \cap A(x, cr, r) = \emptyset$, then $A \subset B(x, cr)$ or $A \cap B(x, r) = \emptyset$. An equivalent definition is the existence of a different but quantitatively related constant $c > 0$ such that

$$\operatorname{cap} A \cap B(x, r) \geq cr \quad \text{for all} \quad x \in A \quad \text{and} \quad r < \operatorname{diam} A.$$

See Exercise IX.3 in [12] for 13 other equivalent definitions.

We will only deal with uniformly perfect subsets of the real line. There is an interesting connection between uniformly perfect sets and John domains due to Andrievskii [3]: If $A \subset [-1, 1]$ has positive capacity, then $f = e^{g_A + ig^*}$ is a conformal map of $\mathbb{C} \setminus [-1, 1]$, where g^* is a harmonic conjugate of g_A, chosen to be real on $[1, \infty)$ such that $\overline{f(\overline{z})} = f(z)$. The complement of $D = f(\mathbb{C} \setminus [-1, 1])$ consists of the closed unit disc, together with a collection of pairs of radial segments $[e^{it_j}, r_j e^{it_j}] \cup [e^{-it_j}, r_j e^{-it_j}]$. Each pair corresponds to a component interval of $[-1, 1] \setminus A$. With this setup, Andrievskii has shown

Theorem 2.9 (Andrievskii, [3])**.** *The set $A \subset [-1, 1]$ is uniformly perfect if and only if the domain $f(\mathbb{C} \setminus [-1, 1])$ is a John domain.*

His proof shows that this is quantitative, namely the uniform perfectness constant and the John constant depend only on each other.

In Section 4 we will need the technical Proposition 2.12. We place it here because it fits logically, would disrupt the main proof later, and also because a simple proof can be based on Andrievskii's theorem. But since our proof uses the Lemma 3.9 below (only the special case $N = 1$ is needed here), the cautious reader could skip ahead and read Section 3 first. Or even better just skip ahead to Section 3, and return later. The first lemma roughly says that no matter how a uniformly perfect set U is partitioned by disjoint intervals, the capacity density is large "in many places".

Lemma 2.10. *For every c there exist c', γ such that the following holds: If $A \subset [a, b]$ is c-uniformly perfect with $a \in A$ and $b \in A$, if $[a, b] = \cup_j I_j$ is a partition into intervals with disjoint interiors, and if*

$$\mathcal{J} = \{j : \operatorname{cap}(I_j \cap A) \geq \gamma |I_j|\},$$

then

$$A \cap \bigcup_{j \in \mathcal{J}} I_j$$

contains a $c'-$uniformly perfect set of diameter $\geq c'|b - a|$, and consequently is of capacity comparable to $|b - a|$.

The sets $A_n = \cup_{j=1}^{n} [j/n, j/n + 1/n^2]$ and the partitions $I_{j,n} = [j/n, j + 1/n]$ show that the assumption of uniform perfectness is essential: Indeed, the capacity of A_n is bounded away from zero, while the capacity density in each $I_{j,n}$ is $1/n$.

Proof. We may assume that $[a,b]=[-1,1]$ and consider the conformal map f and domain D of Andrievskii's theorem. Denote rI_j the interval with same center as I_j and length $|rI_j| = r|I_j|$. If $(1-\varepsilon)I_j \cap A \neq \emptyset$, then $\operatorname{cap}(I_j\cap A) \geq c\frac{\varepsilon}{2}|I_j|$. Consequently, if $\operatorname{cap}(I_j \cap A) < c\frac{\varepsilon}{2}|I_j|$, then $(1-\varepsilon)I_j$ is contained in a component $\hat{I}_j$ of $[-1,1]\setminus A$, and by quasisymmetry of f in the internal metric, Theorem 2.3, we have $\operatorname{diam} f(I_j)\cap\mathbb{T} < \delta(r_j-1)$. Thus $f(I_j)\cap\mathbb{T}$ is contained in the "shadow" of size $\delta(r_j-1)$ cast by the "trees" $[e^{it_j}, r_je^{it_j}]$ and $[e^{-it_j}, r_je^{-it_j}]$. Since D is a John-domain, the set $S=\{(t_j, r_j-1)\}$ satisfies the assumptions of the "sunny Lemma" 3.9, and for ε small enough there is a $\hat{c}-$uniformly perfect set $F\subset\mathbb{T}$ contained in

$$\mathbb{T}\setminus\bigcup_j [e^{\pm t_j-\delta\ell_j}, e^{\pm t_j+\delta\ell_j}] \quad , \quad \ell_j = r_j-1.$$

The internal distance of D and the euclidean distance are comparable on F (this is easy to prove directly using the special form of D, but can also be proven in the same way as Lemma 3.10 below). Thus the preimage $f^{-1}(F)$ is uniformly perfect and of large diameter, hence of capacity bounded away from 0. □

The next lemma goes in the opposite direction. It says that if a set of small capacity is enlarged by replacing regions of relatively large density with intervals, then the capacity remains small.

Lemma 2.11. *For every $\varepsilon>0$ and $\eta>0$ there is δ such that the following is true: If $E\subset[0,1]$ has*

$$\operatorname{cap} E \leq \delta$$

and if

$$\mathcal{J} = \{j : \operatorname{cap}(I_j\cap E) \geq \eta|I_j|\},$$

then

$$\operatorname{cap}\Big(\bigcup_{j\in\mathcal{J}} I_j \cup E\Big) \leq \varepsilon.$$

Proof. Approximate E by a compact subset A such that $A_j = I_j\cap A$ satisfies $\operatorname{cap} A_j \geq \eta/2|I_j|$ for $j\in\mathcal{J}$. If D_j denotes the disc of radius $|I_j|$ centered at the midpoint of I_j, then $\omega_z(A_j, 2D_j\setminus A_j) \gtrsim 1/\log(1/c)$ for $z\in\partial D_j$ by (2.3) and hence $\omega_z(A_j, D_2(0)\setminus A_j)\geq c'$ for $z\in\partial D_j$ by the maximum principle. It follows that

$$\frac{1}{c'}\omega_z(\mathcal{A}, D_2(0)\setminus\mathcal{A}) \geq 1 \geq \omega_z(\mathcal{I}, D_2(0)\setminus\mathcal{I}),$$

for $z\in\cup_{j\in\mathcal{J}}\partial D_j$, where $\mathcal{A}=\cup_{j\in\mathcal{J}}A_j$ and $\mathcal{I}=\cup_{j\in\mathcal{J}}I_j$. By the maximum principle, the same inequality holds for $|z|=1$, and the lemma follows from (2.3) together with the subadditivity of $1/\log(1/\operatorname{cap})$. □

The conclusion of Lemma 2.10 regarding capacity remains true if a set of small capacity E is removed from U. The following generalization is taylored to our application in the proof of Theorem 1.1. Roughly speaking, we show that there is always a large capacity density simultaneously somewhere in the domain and the range of a quasisymmetric map of a uniformly perfect set.

Proposition 2.12. *For every $c > 0$ and $K > 0$ there are $\mu, \nu > 0$ such that the following holds: If $U, U' \subset [0,1]$ are $c-$uniformly perfect sets of diameter $\geq c$, if ϕ is a $K-$quasisymmetric automorphism of $[0,1]$, if*

$$[0,1] = \bigcup_r J_r = \bigcup_r J'_r$$

are two partitions into disjoint intervals such that

$$\phi(U \cap J_r) = U' \cap J'_r$$

for all r, and if $E, E' \subset [0,1]$ have $\operatorname{cap} E < \mu, \operatorname{cap} E' < \mu$, then there is an index r such that

$$\operatorname{cap}(J_r \cap U \setminus (E \cup \phi^{-1}(E')) \geq \nu |J_r|$$

and

$$\operatorname{cap}(J'_r \cap U' \setminus (E' \cup \phi(E)) \geq \nu |J'_r|.$$

Notice that we do not assume the endpoints of the intervals J_r to be in U or to be mapped to the endpoints of J'_r. Under such an assumption, the capacity density of U in J_r would be large if and only if the density of U' in J'_r is large and the proof would simplify.

Proof. At the expense of different constants we may assume $0, 1 \in U \cap U'$. Apply Lemma 2.10 with $A = U$ and obtain constants c', γ' and a $c'-$uniformly perfect set $\hat{U} \subset U$ of diameter $\geq c'$ such that with

$$\mathcal{J} = \{r : \operatorname{cap}(J_r \cap U) \geq \gamma' |J_r|\}$$

we have

$$\hat{U} \subset U \cap \bigcup_{r \in \mathcal{J}} J_r.$$

Since ϕ is quasisymmetric, $\phi(\hat{U})$ is uniformly perfect and of large diameter as well. Again by Lemma 2.10 there are c'' and γ'' depending only on c and K such that with

$$\hat{\mathcal{J}} = \{r : \operatorname{cap}(J'_r \cap \phi(\hat{U})) \geq \gamma' |J'_r|\}$$

the sets

$$\hat{U}' = \phi(\hat{U}) \cap \bigcup_{r \in \hat{\mathcal{J}}} J'_r \quad \text{and} \quad \phi^{-1}(\hat{U}')$$

have large capacity too, $\min(\mathrm{cap}\,\hat{U}', \mathrm{cap}\,\hat{U}, \mathrm{cap}\,\phi^{-1}(\hat{U}')) \geq c''$. Here we have used the fact that quasiconformal maps quasi-preserve capacity [1]. Notice $\hat{\mathcal{J}} \subset \mathcal{J}$ and for every $r \in \hat{\mathcal{J}}$, both U and U' have large capacity density

$$\geq \gamma = \min(\gamma', \gamma'')$$

in J_r and J_r'. To deal with the exceptional sets E and E', let

$$\mathcal{I} = \{r : \mathrm{cap}(J_r \cap (E \cup \phi^{-1}(E'))) > \frac{\gamma}{2}|J_r|\}, \quad \mathcal{I}' = \{r : \mathrm{cap}(J_r' \cap (E' \cup \phi(E))) > \frac{\gamma}{2}|J_r'|\}$$

and consider

$$\overline{E} = E \cup \phi^{-1}(E') \cup \bigcup_{r \in \mathcal{I}} J_r \quad , \quad \overline{E}' = E' \cup \phi(E) \cup \bigcup_{r \in \mathcal{I}'} J_r'.$$

By subadditivity and quasiconformality, the sets $E \cup \phi^{-1}(E')$ and $E' \cup \phi(E)$ have small capacity δ controlled by μ and K. By Lemma 2.11 we can choose μ so small that $\mathrm{cap}\,\overline{E} \cup \phi^{-1}(\overline{E}') < c''/2$. Consequently

$$\mathrm{cap}\,\phi^{-1}(\hat{U}') \setminus (\overline{E} \cup \phi^{-1}(\overline{E}')) > 0$$

and in particular this set is non-empty. Any r for which J_r has non-empty intersection with this set will satisfy the conclusion of the proposition: Indeed, if $J_r \cap \phi^{-1}(\hat{U}') \neq \emptyset$, then $r \in \mathcal{J} \setminus \mathcal{I}$ so that the density of $U \setminus (E \cup \phi^{-1}(E'))$ in J_r is bounded from below by subadditivity, and similarly $r \in \hat{\mathcal{J}} \setminus \mathcal{I}'$ which implies that the density of $U' \setminus (E' \cup \phi(E))$ in J_r' is bounded from below. $\square$

3 Gehring trees have quasisymmetric weldings

3.1 Conformal rectangles and Glueing along Cantor sets

In this section we will describe one of the central ideas of the paper, namely a condition on the glueing of two rectangles under which the conformal modulus stays controlled.

Definition 3.1. A $C-$*rectangle* is a triple (D, A, B), where $D = [0, X] \times [0, 1]$ is a rectangle with $1/C \leq X \leq C$, where A (resp. B) is a $C-$uniformly perfect subset of the left (resp. right) boundary $\{0\} \times [0, 1]$ (resp. $\{X\} \times [0, 1]$), and where $\mathrm{diam}\, A \geq 1/C$ and $\mathrm{diam}\, B \geq 1/C$. Any conformal image onto a simply connected domain is called a *conformal* $C-$*rectangle*, where now A and B have to be interpreted as sets of prime ends.

Consider two rectangles $D_1 = [0, X_1] \times [0, 1]$ and $D_2 = [0, X_2] \times [0, 1]$ together with their left boundaries $L_j = \{0\} \times [0, 1]$ and their right boundaries $R_j = \{X_j\} \times [0, 1], j = 1, 2.$

It is well-known that, if φ is a quasisymmetric homeomorphism between R_1 and L_2, then *conformal welding* of D_1 and D_2 via φ yields a conformal rectangle of modulus bounded in terms of X_1, X_2 and the quasi-symmetry constant. More precisely, there are conformal rectangles D_1', D_2' with disjoint interiors and conformal maps $f_j : D_j \to D_j'$ sending corners to corners such that $f_1 = f_2 \circ \varphi$ on the right boundary R_1, and such that $D_1' \cup D_2'$ is a rectangle $[0, X] \times [0, 1]$ of modulus X between $X_1 + X_2$ and $M(X_1, X_2, K)$, see Figure 3.1.

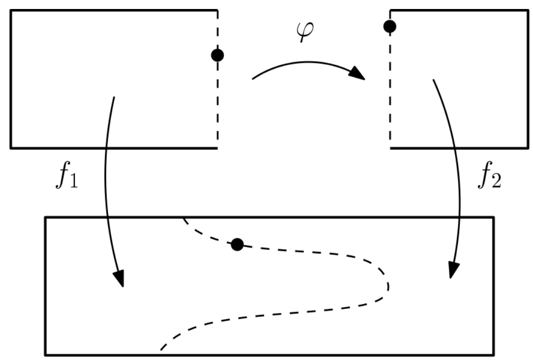


Figure 3.1: Conformal welding of two rectangles

The following generalization shows that one can give up knowledge of the welding homeomorphism, as long as one has control on a sufficiently large subset.

Proposition 3.2. *If (D_1, A_1, B_1) and (D_2, A_2, B_2) are $C-$rectangles and if $\varphi : B_1 \to A_2$ is an increasing $K-$quasisymmetric homeomorphism, then for any homeomorphic extension $\Phi : R_1 \to L_2$ of φ that admits conformal welding, the welded region is a $C'-$rectangle with constant only depending on C and K.*

Notice that the resulting modulus only depends on φ and not on the particular extension Φ. The proposition serves the purpose of illustrating one of the key points of our approach, but will not be used directly in the proof of our main result. The observant reader will notice that we only use much weaker assumptions, namely that the capacities of B_1 and A_2 are bounded away from zero, and that ϕ maps sets of small capacity to sets of small capacity.

Proof. Since B_1 and A_2 are uniformly perfect, it is easy to see that φ has a quasisymmetric extension Φ_0 to R_1 and therefore is Hölder continuous. Let ρ be admissable for the family of curves that join the vertical sides of the rectangle $D_1' \cup D_2'$, and let

$$\rho_1(z) = \rho(f_1(z))|f_1'(z)| \quad , \quad \rho_2(z) = \rho(f_2(z))|f_2'(z)|$$

denote the pullbacks of $\rho|_{D_i}$. We wish to find a lower bound for

$$\int_{D_1' \cup D_2'} \rho^2 dxdy = \int_{D_1} \rho_1^2 dxdy + \int_{D_2} \rho_2^2 dxdy.$$

Let ε be small and suppose that $\int_{D_1} \rho_1^2 dxdy < \varepsilon^3$. Let E_1 be the set of those points $x \in R_1$ for which there does not exist a curve γ of ρ_1-length $< \varepsilon$ joining L_1 to x. Then ρ_1/ε is admissable for the family Γ_{E_1} of curves joining L_1 to E_1 in D_1, and it follows that $M(\Gamma_{E_1}) < \varepsilon^3/\varepsilon^2 = \varepsilon$. By Pfluger's theorem, $\operatorname{cap} E_1 \lesssim e^{\pi/\varepsilon}$. By the Hölder continuity of φ, the capacity of $\varphi(E_1)$ is small too. Similarly, the set $E_2 \subset L_2$ of points that cannot be joined to R_2 by a ρ_2 short curve has small capacity if $\int_{D_2} \rho_2^2 dxdy < \varepsilon^3$, and again by Hölder continuity, $cp\varphi^{-1}(E_2)$ is small too. By subadditivity of capacity, $B_1 \setminus (E_1 \cup \varphi^{-1}(E_2))$ has positive capacity and hence is non-empty if ε is small enough. If $x \in B_1 \setminus (E_1 \cup \varphi^{-1}(E_2))$, there are curves γ_1 joining L_1 to x and γ_2 joining R_2 to $\varphi(x)$, both of ρ_i-length $< \varepsilon$. Consequently the curve $\gamma = f_1(\gamma_1) \cup f_2(\gamma_2)$ has $\rho-$length < 1 is $\varepsilon < 1/2$, and ρ is not admissable. $\square$

For later use, we record the following technical lemma. Roughly speaking it says that in *every* conformal metric on (D, A, B) most points of A can be joined to most points of B by a short curve. Consider a rectangle $D = [0, X] \times [0, 1]$ with $1/C \leq X \leq C$ and again denote $L = \{0\} \times [0, 1]$ the left and $R = \{X\} \times [0, 1]$ the right boundary.

Lemma 3.3. *If $\rho_0 \geq 0$ is a measurable function on D of ρ_0-area ≤ 1,*

$$\int_D \rho_0^2 dxdy \leq 1,$$

then for every $\lambda > 0$ there are subsets $E_L \subset L$ and $E_R \subset R$ with

$$\operatorname{cap}(E_L) \leq C' e^{-\pi\lambda^2} \quad , \quad \operatorname{cap}(E_R) \leq C' e^{-\pi\lambda^2} \tag{3.1}$$

such that for every $x \in L \setminus E_L$ and every $y \in R \setminus E_R$ there is a curve $\gamma_{x,y} \subset D$ with endpoints x and y of bounded ρ-length

$$\int_{\gamma_{x,y}} \rho_0 |dz| \leq 2\lambda,$$

where C' depends only on C.

Proof. Decompose the top edge as $[0, X] \times \{1\} = T_L \cup T_R$ with $T_L = [0, X/2] \times \{1\}$ and $T_R = [X/2, 1] \times \{1\}$. Denote E_L the set of those $x \in L$ for which there does not exist a single curve $\gamma \subset D$ of $\rho-$length $\leq \lambda$ joining x to T_R, and similarly denote E_R denote the set of those $y \in R$ without a curve of $\rho-$length $\leq \lambda$ joining y and T_L. If Γ denotes the family of all curves $\gamma \subset D$ that join a point of E_L to T_R, then $\rho_0 = \rho/\lambda$ is admissable for Γ and it follows that the modulus is bounded by

$$M(\Gamma) \leq \frac{1}{\lambda^2}.$$

By Pfluger's theorem (2.4) we have $\operatorname{cap} E_L \leq C' e^{-\pi\lambda^2}$, and similarly $\operatorname{cap} E_R \leq C' e^{-\pi\lambda^2}$. Now if $x \in L \setminus E_L$ and $y \in R \setminus E_R$, then there are curves γ_x from x to T_R and γ_y from y to T_L of $\rho-$length bounded by λ, and since these curves have to intersect there exists a curve $\gamma_{x,y} \subset \gamma_x \cup \gamma_y$ joining x and y of length $\leq 2\lambda$. $\square$

The same proof applies to the following variant. We leave the details to the reader.

Lemma 3.4. *If $\rho_0 \geq 0$ is a measurable function on $\mathbb{D}$ of ρ_0-area ≤ 1, then for every $\lambda > 0$ there are subsets $E \subset \partial\mathbb{D} \cap \mathbb{H}_+$ and $E' \subset \partial\mathbb{D} \cap \mathbb{H}_-$ with*

$$\operatorname{cap}(E) \leq C' e^{-C''\lambda^2} \quad , \quad \operatorname{cap}(E') \leq C' e^{-C''\lambda^2} \tag{3.2}$$

such that for every $x \in \partial\mathbb{D} \setminus E$ and every $y \in \partial\mathbb{D} \setminus E'$ there is a curve $\gamma_{x,y} \subset \mathbb{D}$ with endpoints x and y of bounded ρ-length

$$\int_{\gamma_{x,y}} \rho_0 |dz| \leq \lambda,$$

where C' and C'' are absolute constants.

3.2 Localization of Gehring trees

The main result of this section is the following decomposition of annuli centered at a Gehring tree into conformal rectangles that are cyclically glued as in Proposition 3.2.

Proposition 3.5. *For every Gehring tree T there are constants K_1, N, M and C (depending only on $K(T)$) such that for every $p \in T$ and every $0 < r < \operatorname{diam} T/C$ there are disjoint conformal $(M, C)-$rectangles $(D_i, A_i, B_i), i = 1, 2, ..., n$ where*

$$n \leq N, \tag{3.3}$$

$$D_i \subset (\overline{\mathbb{C}} \setminus T) \cap \{z : r < |z - p| < Cr\}, \tag{3.4}$$

$$A_i, B_i \subset T, \text{ and } A_{i+1} = B_i \tag{3.5}$$

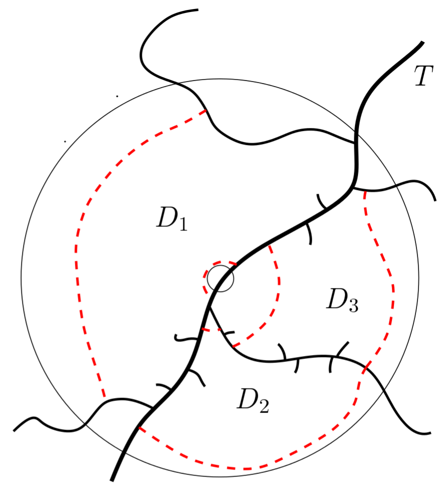
$$\text{the “horizontal sides” of the } D_i \text{ are geodesics of } \overline{\mathbb{C}} \setminus T, \tag{3.6}$$

$$\text{The set } \cup_{i=1}^n \overline{D}_i \text{ separates } 0 \text{ and } \infty. \tag{3.7}$$

Moreover, for every i, if (D'_i, A'_i, B'_i) denotes a rectangle conformally equivalent to (D_i, A_i, B_i), then

$$\text{there is a } K_1 - \text{quasisymmetric homeomorphism between } B'_i \text{ and } A'_{i+1}. \tag{3.8}$$

Notice that the index $i+1$ in (3.5) and (3.8) has to be interpreted $\mod n$, so that D_n glues back to D_1. In particular, if $n = 1$, the two vertical sides of D_1 are glued together.



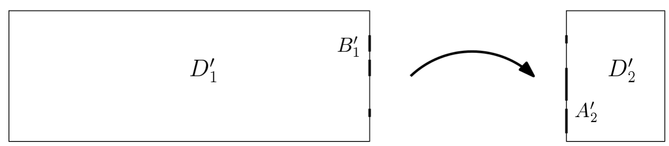


Figure 3.2: The conformal rectangles D_i of Proposition 3.5. The dashed boundary components are hyperbolic geodesics of $\hat{\mathbb{C}} \setminus T$ and correspond to the horizontal boundary of the rectangles.

The proof of Proposition 3.5 requires some preparation. Assume $p = 0 \in T$, $\operatorname{diam} T > R$, and denote $A(r, R) = \{r < |z| < R\}$. Fix a connected component D of $A(r, R) \setminus T$. We say that D *crosses* $A(r, R)$ if $\overline{D} \cap C_r$ and $\overline{D} \cap C_R$ contain non-trivial intervals. The boundary of D contains two arcs $\partial_\ell D$ and $\partial_h D$ of T joining C_r and C_R, where ℓ stands for "lower" and h for "higher" in logarithmic coordinates. They can be defined formally as follows: Since $\operatorname{diam} T > R$, the closed set $T \cup C_r \cup C_R$ is connected and D is simply connected, so that a continuous branch of $\log z$ is well-defined on D. There is an arc $\gamma' = \log \gamma$ in $\log D$ joining the lines $x = \log r$ and $x = \log R$. The set $\{\log r < x < \log R\} \setminus \log D$ has two unbounded components. Both unbounded components have boundary consisting of a crosscut σ together with two half-infinite vertical lines. Denote σ_ℓ resp. σ_h the lower resp. higher of these two crosscuts (they are disjoint because they belong to different complementary components of γ'). Finally denote $\partial_\ell D$ and $\partial_h D$ the images of σ_ℓ and σ_h under the exponential function. Note that they can be disjoint, identical, or neither.

The following lemma is the key to the inductive construction of the conformal rectangles. Consider three disjoint curves γ_j that crosses and annulus $A(r_1, r_2)$. We say that γ_1 lies *between* γ_2 and γ_3 if the endpoints of $\gamma_2, \gamma_1, \gamma_3$ are positively oriented on the circle. Thus γ_1 either crosses between γ_2 and γ_3, or between γ_3 and γ_2, but not both. We use the same terminology for connected sets.

Lemma 3.6. *For every $M > 0$ there exists a constant $C = C(M, K(T))$ such that the following holds: If D_1 and D_2 are not necessarily distinct components of $A(r,R) \setminus T$ that cross $A(r,R)$ and if $\partial_h D_1 \cap \partial_\ell D_2$ does not contain an arc of diameter $> Mr$, then there exists a component D_3 crossing $A(Cr,R)$ between $\partial_h D_1$ and $\partial_\ell D_2$.*

Proof. If $\partial_h D_1 \cap \partial_\ell D_2$ does not contain an arc of diameter $> Mr$, then $\partial_h D_1$ and $\partial_\ell D_2$ are disjoint in $A(Cr,R)$ for sufficiently large C : Suppose not, then there would be a point $x \in \partial_h D_1 \cap \partial_\ell D_2 \cap A(Cr,R)$. Consider the points $x_h \in \partial_h D_1 \cap C_r$, $x_\ell \in \partial_\ell D_2 \cap C_r$ and the "center" x' of the "triangle" on T with vertices x_ℓ, x_h and x,

$$x' = T(x_\ell, x_h) \cap T(x_\ell, x) \cap T(x_h, x).$$

Since $T(x,x') \subset \partial_h D_1 \cap \partial_\ell D_2$, it follows that $\operatorname{diam} T(x,x') \le Mr$, hence $|x'| > (C-M)r$. Thus

$$\operatorname{diam} T(x_\ell, x_h) > (C-M-1)r \ge \frac{C-M-1}{2}|x_\ell - x_h|,$$

contradicting the quasiarc property 2.2 if $(C-M-1)/2 > K$.

The same argument shows that there is no curve $\sigma \subset T$ joining $\partial_h D_1$ and $\partial_\ell D_2$ in $\cap A(Cr,R)$. Hence there is a curve $\gamma \subset A(Cr,R) \setminus (T \cup D_1 \cup D_2)$ between $\partial_h D_1$ and $\partial_\ell D_2$ from C_{Cr} to C_R. The component of $A(Cr,R) \setminus T$ containing γ is the required D_3. □

We will need the following bound on the number of components that can cross an annulus.

Lemma 3.7. *If $R/r \ge 4K$, then the number of components of $A(r,R) \setminus T$ that cross $A(r,R)$ is bounded above by $8K\pi$.*

Proof. Denote $D_1, ..., D_n$ the components that cross $A(r,R)$. The arcs $\partial_\ell D_i$ meet C_r resp. C_R in points $a_{\ell,i}$ resp. $b_{\ell,i}$ and similarly $\partial_h D_i$ meet C_r resp. C_R in points $a_{h,i}$ resp. $b_{h,i}$. For each i, one of the two arcs of $C_R \setminus b_{\ell,i}, b_{h,i}$ belongs to $\overline{D_i}$ and thus is essentially disjoint from all the other $\overline{D_j}$. Thus there are n disjoint arcs of C_R bounded by pairs $b_{\ell,i}, b_{h,i}$. Consequently, if $d_j = |b_{\ell,j} - b_{h,j}|$ is minimal among $d_i, 1 \le i \le n$, then $d_j \le 2\pi R/n$. By the Ahlfors condition (2.2),

$$\operatorname{diam} T(b_{\ell,j}, b_{h,j}) \le K\frac{2\pi R}{n}.$$

In particular, the arc $\gamma \subset T$ that joins $\partial_\ell D_j$ and $\partial_h D_j$ is of distance $\le K\frac{2\pi R}{n}$ from C_R, hence of distance

$$d \ge R - r - K\frac{2\pi R}{n}$$

from C_r. Since $a_{\ell,j}$ and $a_{h,j}$ connect through the same arc γ, we have

$$\operatorname{diam} T(a_{\ell,j}, a_{h,j}) \ge d \ge \frac{R}{2}$$

if $R > 4r$ and $n > 8\pi K$. On the other hand,

$$\operatorname{diam} T(a_{\ell,j}, a_{h,j}) \leq K|a_{\ell,j} - a_{h,j}| \leq K2r < \frac{R}{2}$$

if $R > 4Kr$, and we conclude that $n \leq 8\pi K$ whenever $R/r > 4K$. □

Next, still assuming that $0 \in T$ we will construct a chain of domains D_i that already has most of the desired features of Proposition 3.5, namely (3.3),(3.4), (3.6) and (3.7).

Lemma 3.8. *With C and M as in Lemma 3.6, there are constants N, C_1, C_2 such that whenever $R = C_2 r <$ diam T there are domains $D_1, D_2, ..., D_n$ in $A(r, R)$ with $n \leq N$ that are all crossing $A(C_1 r, R)$ and satisfy the following: Each D_i is a connected component of $A(C^{n_i} r, R) \setminus T$ for some $0 \leq n_i \leq N$, each intersection $\partial_h D_i \cap \partial_\ell D_{i+1}$ contains an arc α_i of diameter $\geq M \max(C^{n_i} r, C^{n_{i+1}} r)$, and $\cup_i \overline{D_i}$ separates $A(r, R)$.*

Proof. Set $C_1 = C^{8\pi K}$ where C is the constant of Lemma 3.6, and $C_2 = 4KC_1$. Fix a component D_1 crossing $A(r, R)$. If $\partial_h D_1 \cap \partial_\ell D_1$ contains an arc of diameter $\geq Mr$, set $N = 1$ and we are done. Else, by Lemma 3.6, there is a component D_2 crossing $A(Cr, R)$ between $\partial_h D_1$ and $\partial_\ell D_1$. Note that D_1 also crosses $A(Cr, R)$. If the pair D_1, D_2 does not satisfy the claim, repeated application of Lemma 3.6 (relabeling the domains if necessary to keep the cyclic order) yields a sequence of disjoint domains $D_1, ..., D_n$ that all cross $A(C^{n-1} r, R)$. By Lemma 3.7, this process has to stop when $n \leq N = 8K\pi$. Joining a point of $\partial_h D_{i-1} \cap \partial_\ell D_i$ with a point of $\partial_h D_i \cap \partial_\ell D_{i+1}$ by a Jordan arc in D_i, we obtain a Jordan curve in $\cup_i \overline{D_i}$ separating C_r and C_R. □

Next, we slightly modify the D_i to turn them into conformal rectangles of controlled modulus with geodesic boundaries. Roughly speaking, since we have no lower bound for the size of the "inner boundary" $\partial D_i \cap C_r$ of D_i, we just replace it by a larger arc: By construction, each D_i is a connected component of $A(C^{n_i} r, R) \setminus T$. Let $D_i' \subset D_i$ be the component of $A((C^{n_i} + 1)r, R) \setminus T$ that joins the two boundary circles, and let I_1 be the arc on $C_{(C^{n_i}+1)r}$ that separates $C_{C^{n_i} r}$ and C_R. By the Ahlfors condition, the length of I_1 is comparable to $C^{n_i} r$. Let γ_1 be the hyperbolic geodesic of $\overline{\mathbb{C}} \setminus T$ joining the two endpoints $a_{1,i}, b_{1,i}$ of I_1 (viewed as prime ends). Then the diameter of γ_1 is comparable to $C^{n_i} r$ (the upper bound follows from the Gehring-Hayman inequality, the lower bound from the Ahlfors-condition),

$$\operatorname{diam} \gamma_1 \leq C_0 C^{n_i} r. \tag{3.9}$$

Furthermore, the distance of γ_1 to 0 is comparable to $C^{n_i} r$ and hence greater than a constant times r. Similarly, let I_2 be the arc on S_R that separates $S_{C^{n_i} r}$ and ∞, and let γ_2 be the geodesic joining the endpoints $a_{2,i}, b_{2,i}$ of I_2. The diameter of γ_2 is comparable to R, again by the Gehring-Hayman inequality. Now let D_i' be the connected component of $\overline{\mathbb{C}} \setminus T$ bounded by γ_1 and γ_2. These are the domains of Proposition 3.5. Notice that properties (3.3),(3.4), (3.6) and (3.7) are satisfied by Lemma 3.8. Notice also that the set

$$\alpha_i \setminus D(0, (C^{n_i} + 1)r)$$

contains an arc α_i' of diameter comparable to r,

$$\operatorname{diam} \alpha_i' \geq (MC^{n_i} r - 2(C^{n_i} + 1)r)/2,$$

and that this arc is contained in $\partial_h D_i' \cap \partial_\ell D_{i+1}'$.

To simplify notation, from now on we will drop the prime and will simply write D_i and α_i instead of D_i' and α_i'. Assume without loss of generality counterclockwise ordering of a_1, b_1, b_2, a_2, and notice that the conformal modulus of the topological rectangles $(D_i, a_1, b_1, b_2, a_2)$ is bounded above and below : This can easily be seen using Rengel's inequality

$$\frac{w^2}{\text{area} D} \leq M \leq \frac{\text{area} D}{h^2},$$

where w and h are the distances between the opposite pairs of sides. Indeed, the sides (a_1, b_1) and (b_2, a_2) trivially have distance of order $R - C^{n_i} r$, and the sides (b_1, b_2) and (a_2, a_1) have distance of order $C^{n_i} r$ again by the Ahlfors condition. The area is of order R^2, since the diameter of γ_2 is of order R.

We will now turn to the construction of the sets A_i, B_i. Since D_i and D_{i+1} connect through the arc $\alpha_i \subset T$, we will define B_i as a subset of α_i. It can be shown that $(D_i, \alpha_{i-1}, \alpha_i)$ is a conformal (M, C)-rectangle for suitable constants (only the uniform perfectness of the image of α under a conformal map onto a rectangle would require proof). But this is NOT the appropriate choice of B_i: Indeed, every triple point on α corresponds to two prime ends on one side and one prime end on the other side of α, so that the choice $B_i = A_{i+1} = \alpha$ does not even allow for a bijection between the corresponding sets in rectangular coordinates, see Figure 3.2 . To obtain a homeomorphism, we need to stay away from the branch points of T. Roughly speaking, to obtain a quasisymmetric homeomorphism, we need to stay away from the branch points in the following quantitative way: If β is a branch of T branching off a point x in the interior of α (not one of the two endpoints), then we need to remove an interval of size proportional to $\operatorname{diam} \beta$ from α. More precisely, since α is a quasiconformal arc, there is a global quasiconformal map F that sends α to the interval $[0, 1]$ on the real line. For every component β of $T \setminus \alpha$ with $\overline{\beta} \cap \alpha \neq \emptyset$, the intersection consists of a single point x_β. Denote

$$\ell_\beta = \operatorname{diam} F(\beta).$$

We will show that for sufficiently small $\delta > 0$, the set

$$B_i = A_{i+1} = F^{-1}\Big(\, [0,1] \setminus \bigcup_\beta [F(x_\beta) - \delta\ell_\beta, F(x_\beta) + \delta\ell_\beta] \, \Big) \tag{3.10}$$

satisfies the requirements of Proposition 3.5. We begin by describing a set of assumptions guaranteeing that such a set it non-empty. A picturesque description of the situation is as follows: Suppose trees (vertical line segments of length ℓ_β) grow from the forest floor (the interval $[0,1]$) in such a way that they are not too close to each other: The distance between two trees is bounded below by a constant times the height of the smaller tree. Each tree casts a shadow of size proportional to its height. Then there are many sunny places on the forest floor (assuming this proportionality constant is small). More generally (corresponding to branch points of order more than three) we need to allow for a bounded number of trees to get close. A precise statement is the

Sunny Lemma 3.9. *For every integer $N \geq 1$ and real $M > 0$ there are $\delta = \delta(N, M)$ and $c = c(N, M)$ such that the following holds: Suppose $S = \{(x_n, \ell_n)\}$ is a collection of pairs $(x, \ell) \in [0,1] \times (0, M]$ with the property that for every interval $I = [a, b] \subset [0,1]$, the number of pairs $(x, \ell) \in S$ with $x \in [a, b]$ and $\ell \geq M|b - a|$ is $\leq N$. Then*

$$[0,1] \setminus \bigcup_{(x,\ell)\in S} [x - \delta\ell, x + \delta\ell]$$

contains a $c-$uniformly perfect set of diameter at least $1/2$.

Proof. Set

$$r = \frac{1}{4N + 4}$$

and fix

$$\delta < \frac{1}{2(4N + 4)M}.$$

Beginning with $\mathcal{L}_0 = \{[0,1]\}$, inductively construct a nested collection $\mathcal{L}_n$ of disjoint "potentially sunny intervals" of size r^n as follows: Fix $I \in \mathcal{L}_n$ and subdivide I into $4N + 4$ equal sized intervals of size r^{n+1}. By assumption, there are at most N intervals ("shadows") $[x - \delta\ell, x + \delta\ell]$ for which $x \in I$ and $Mr^n < \ell \leq Mr^{n-1}$. By the definition of δ, each of these shadows $[x - \delta\ell, x + \delta\ell]$ will intersect at most two (adjacent) of the $4N + 4$ intervals. In addition, there are at most two more of the $4N + 4$ intervals that intersect a shadow $[x - \delta\ell, x + \delta\ell]$ with $x \notin I$ and $Mr^n < \ell \leq Mr^{n-1}$ (one on either endpoints of I). Hence there are at least $2N + 2$ intervals remaining, and they constitute the collection of the intervals of $\mathcal{L}_{n+1}$ that are in I. In particular, it follows that

$$\operatorname{diam} \bigcup_{J\in\mathcal{L}_{n+1}, J\subset I} \geq 1/2 \operatorname{diam} I. \tag{3.11}$$

By construction

$$A = \bigcap_n \bigcup_{I\in\mathcal{L}_n} I \quad \subset \quad [0,1] \setminus \bigcup_{(x,\ell)\in S} [x - \delta\ell, x + \delta\ell],$$

and it is easy to see that A is uniformly perfect: If $x \in A$ and if the annulus $A(x; R_1, R_2)$ separates A where $R_1 < R_2 \leq 1$, then consider the minimal n for which $r^n < 4R_1$, and denote $I_n(x)$ the interval of $\mathcal{L}_n$ that contains x. By (3.11), the interval $I_{n-1}(x)$ contains a second interval $J \in \mathcal{L}_n$ of distance more than $1/4r^{n-1}$ and less than r^{n-1} from $I_n(x)$. Thus $A(x; R_1, R_2)$ separates $I \cap A$ and $J \cap A$, which yields an upper bound on R_2/R_1. $\square$

Returning to the definition (3.10) of B_i, we claim that the collection $\{(F(x_\beta), \ell_\beta)\}$ satisfies the assumptions of Lemma 3.9, where $\ell_\beta = \operatorname{diam} F(\beta)$. To this end, let us first notice that by the quasisymmetry of F, there is an upper bound $\ell_\beta \leq M$ that only depends on $K(T)$: Indeed, all arcs β are contained in $D_i \cup D_{i+1}$ so that $\operatorname{diam} \beta \leq \operatorname{diam} D_i \cup D_{i+1} \lesssim \operatorname{diam} \alpha$. Next, fix an interval $[a, b] \subset [0, 1]$ and consider an arc β with $F(x_\beta) \in [a, b]$ and $\ell_\beta > M|b-a|$. Then the quasisymmetry of F implies that

$$\operatorname{diam} \beta \geq M' \operatorname{diam} F^{-1}[a, b]$$

where M' is large when M is large, and since T is a Gehring tree there is an upper bound N on the number of such arcs. Thus the assumptions of the sunny Lemma 3.9 are satisfied, and for sufficiently small δ the set $[0, 1] \setminus \bigcup_\beta [F(x_\beta) - \delta\ell_\beta, F(x_\beta) + \delta\ell_\beta]$ is uniformly perfect and of diameter $\geq 1/2$. Consequently the image under F^{-1} is uniformly perfect (quasiconformal maps distort moduli of annuli only boundedly) and of diameter comparable to $\operatorname{diam} \alpha_i$. We will also need the following

Lemma 3.10. *The internal distance d_i of D_i and the euclidean distance are comparable on $A_i \cup B_i$.*

Proof. Let $x, y \in B_i$, viewed as prime ends of D_i. We need to show that $d_i(x, y) \lesssim |x - y|$. Recall the bounded turning property of the John domain $\overline{\mathbb{C}} \setminus T$: By Theorem 6.3 of [31], the smaller of the two arcs of $\partial^* T$ between x and y has diameter comparable to the internal distance between x and y in $\overline{\mathbb{C}} \setminus T$, which is easily seen to be comparable to the internal distance of D_i. This smaller arc is of the form

$$\alpha(x, y) \cup \bigcup_{x_\beta \in \alpha(x,y)} \beta,$$

where the union is over all components β of $T \setminus \alpha$ with $\overline{\beta} \cap \alpha \neq \emptyset$ and $\beta \subset \overline{D}_i$. Since $x \in B_i$ we have

$$|F(x) - F(x_\beta)| \geq \delta \operatorname{diam} F(\beta)$$

so that

$$|x - x_\beta| \geq \delta' \operatorname{diam} \beta$$

by quasisymmetry of F, and similarly for y. It follows that any arc β with x_β between x and y has diameter bounded by a constant times $|x - y|$, and the Lemma follows in this case. If both x and y are in A_i, the proof is the same. In case $x \in B_i$ and $y \in A_i$, both the euclidean and the internal distance of x and y are bounded above and below, and there is nothing to prove. $\square$

Conclusion of the Proof of Proposition 3.5. We have already established the existence of constants K_1, N, M, C and defined the domains D_i and sets $A_i, B_i \subset \partial D_i \cap T$ satisfying (3.3) - (3.7). It only remains to show that the topological rectangles (D_i, A_i, B_i) are conformal $(M, C)-$rectangles, and to verify (3.8). Recall that we already showed that the conformal modulus of the topological rectangle $(D_i, a_1, b_1, b_2, a_2)$ is bounded above and below, where the four marked points (a_1, b_1, b_2, a_2) are the endpoints of α_i and α_{i-1}, namely $\alpha_i = T(a_1, a_2)$ and $\alpha_{i-1} = T(b_1, b_2)$. Since the diameter of B_i is comparable to the diameter of α_i, and the diameter of A_i is comparable to $\operatorname{diam} \alpha_{i-1}$, the same argument using Rengels inequality shows that the conformal modulus of (D_i, A_i, B_i) is bounded above and below (away from zero).

Now let g_i be the conformal map of D_i onto the rectangle $R_i = [0, X_i] \times [0, 1]$ that takes the extreme points of A_i resp B_i to the left resp. right vertices. Since the modulus X_i is bounded above and below, the rectangle is a John domain with bounded constant. By Theorem 7.4 of [31], g is a quasisymmetric map between D_i and R with respect to the internal metric d_i of D_i. It easily follows that $g_i(A_i)$ and $g_i(B_i)$ are uniformly perfect with constant only depending on those of A_i and B_i, and the quasisymmetry data. Finally, the required homeomorphism between $g_i(B_i)$ and $g_{i+1}(A_{i+1})$ is simply given by the restriction of $g_{i+1} \circ g_i$ to B_i, which is quasisymmetric by two applications of the aforementioned Theorem 7.4 of [31], combined with the bilipschitz-continuity of the restriction to B_i of the identity map on $\partial D_i \cap \partial D_{i+1}$ with respect to the internal metrics of D_i and of D_{i+1}, again using Lemma 3.10. $\square$

3.3 Proof of the “only if” part of Theorem 1.1

Most of the work has already been done by proving Proposition 3.5. Let $\mathcal{L}$ be the lamination of a Gehring tree T and $f : \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \overline{\mathbb{C}} \setminus T$ a conformal map fixing ∞. Given $x \in \mathbb{T}$ and a scale $0 < \rho < \rho_0$ (where ρ_0 will be specified later), let I be the arc of $\mathbb{T}$ of length ρ with initial point x. Applying Proposition 3.5 to $p = f(x)$ and

$$r = \frac{\operatorname{diam} f(I)}{C},$$

we obtain conformal (M, C)-rectangles (D_i, A_i, B_i) separating the annulus $A(p; r, Cr)$, arranged in counter-clockwise order. We may assume the labeling is such that the prime end boundary of D_1 contains points of $f(I)$. Recall the definition of the points $a_{1,i}, a_{2,i}, b_{1,i}, b_{2,i}$, introduced during the construction of D_i right after Lemma 3.8, and denote $\hat{a}_{1,i}, \hat{a}_{2,i}, \hat{b}_{1,i}, \hat{b}_{2,i}$ their preimages under f. The preimages

$$\hat{D}_i = \ f^{-1}(D_i)$$

are bounded by the two arcs $[\hat{a}_{1,i}, \hat{a}_{2,i}]$ and $[\hat{b}_{1,i}, \hat{b}_{2,i}]$ on $\mathbb{T}$ together with the hyperbolic geodesics $< \hat{a}_{1,i}, \hat{b}_{1,i} >$ and $< \hat{a}_{2,i}, \hat{b}_{2,i} >$ of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, see Figure 3.3.

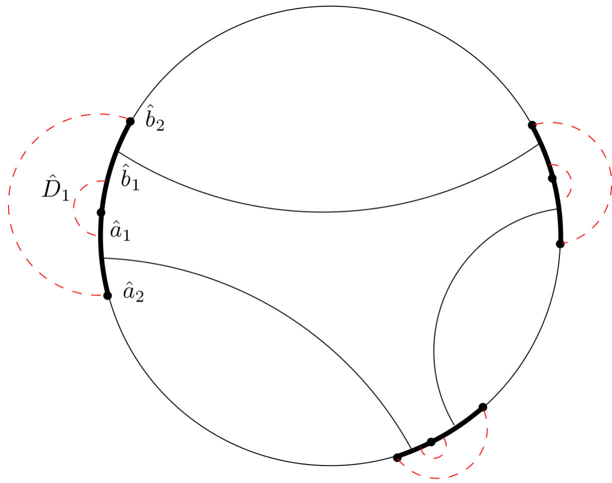


Figure 3.3: bla

Set $p_1 = p$, for each $i = 2, 3, ..., n$ fix any point $p_i \in [a_{1,i}, b_{1,i}] \subset \mathbb{T}$ (for instance $p_i = [a_{1,i}]$), and for $i = 1, 2, ..., n$ let

$$I_i = [b_{2,i}, p_i] \ , \quad I_i' = [p_i, a_{2,i}].$$

Since the conformal modulus of (D, a_1, b_1, b_2, a_2) is bounded above and below for each i, the lengthes $|I_i|$ and I_i' are comparable. Let

$$\hat{A}_i = f^{-1} A_i \cap I_i \ , \quad \hat{A}_i' = f^{-1} B_i \cap I_i'$$

so that for each i, $(\hat{D}, \hat{A}, \hat{A}')$ is a conformal image of (D, A, B) and hence a conformal (M, C)-rectangle. In particular, $\hat{A}$ and $\hat{A}'$ are uniformly perfect and

$$\operatorname{diam} \hat{A}_i \asymp |I_i| \asymp |I_i'| \asymp \hat{\operatorname{diam}} \hat{A}_i'.$$

The conformal maps of $(\hat{D}, \hat{A}, \hat{A}')$ onto a rectangle that send the points a_1, a_2, b_1, b_2 to the corners is quasisymmetric (this is clear since both domains are John with bounded constants, and can be verified directly since it is just a composition of a Mobius transformation with a logarithm). It remains to notice that the quasisymmetric homeomorphisms of (3.8) conjugates to a quasisymmetric homeomorphism between $\hat{A}_i'$ and $\hat{A}_{i+1}$. Thus the sets $\hat{A}_i, \hat{A}_i'$ satisfy all requirements of Theorem 1.1 and we have proved that Gehring trees satisfy the conditions of Theorem 1.1.

4 Quasisymmetric weldings yield Gehring trees

4.1 Finite laminations and balloon animals

Given a finite lamination $\mathcal{L}$, there are a large number of solutions to the realization problem of finding a conformal map f of Δ such that $\mathcal{L}_f = \mathcal{L}$. Koebe [21] showed that laminations without triple points can be realized as conformal laminations of a circle domains ($\overline{\mathbb{C}} \setminus f(\Delta)$ is a union of discs with disjoint interiors). There is nothing really special about discs, and it is feasible that instead of discs one could prescribe any say strictly convex shape (up to homothety). Bishop [7] rediscovered Koebe's theorem and used it to solve a conformal welding problem for Jordan curves. Our approach to solving the welding problem is partly motivated by Bishops important work. However, as we will see later, our method relies on the existence of a realization for which the harmonic measures of complementary components are linearly related. This fails spectacularly for circle domains, as the harmonic measure of a disc is just normalized length measure, while the harmonic measure from the outside at the intersection point of two discs decays exponentially. The purpose of this section is to prove the following proposition, see also Figure 4.1.

Proposition 4.1. *a) For every finite lamination $\mathcal{L}$, there is a simply connected domain $G \subset \overline{\mathbb{C}}$ and a conformal map $f : \Delta \to G$ fixing ∞ such that $\overline{\mathbb{C}} \setminus \overline{G}$ consists of Jordan domains G_i, and such that $f(x) = f(y)$ if and only if $x = y$ or $(x, y) \in \mathcal{L}$.*

b) Moreover, G can be chosen such that there are points $z_i \in G_i$ for which the harmonic measures ω_i of ∂G_i at z_i and ω_∞ of G at ∞ have the property

$$\frac{d\omega_\infty}{d\omega_i} \equiv p_i \quad on \quad \partial G_i \tag{4.1}$$

for all i, where $p_i = \omega(0, \mathbb{D}, \overline{P}_i \cap \partial\mathbb{D}) = \omega_\infty(\partial G_i)$.

c) More generally, given positive numbers $p_{i,j}$ for each boundary arc $\alpha_{i,j}$ of P_i such that $\sum_j p_{i,j} = p_i$, then G and z_i can be chosen such that

$$\frac{d\omega_\infty}{d\omega_i} \equiv p_{i,j} \quad on \quad \beta_{i,j} = f(\alpha_{i,j}) \quad for\ all \quad i, j. \tag{4.2}$$

In b) and c), the domain G and the z_i are unique up to normalization by a linear map $z \mapsto az + b$.

Remark. The domain in part a) of the proposition is highly non-unique. It can for instance be chosen in such a way that each G_i is a disc, and there is only one such collection of discs up to normalization. See [21], and also Lemma 19 in [7] for a closely related circle chain.

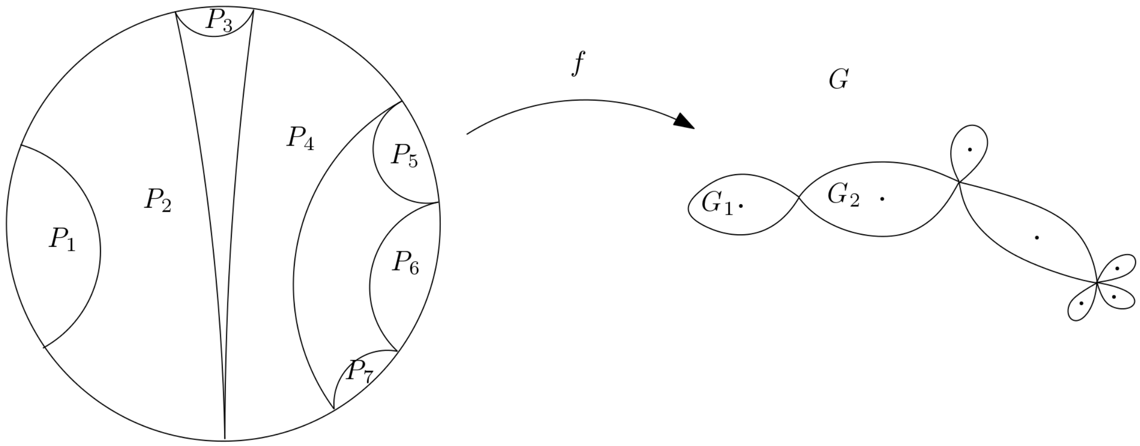


Figure 4.1: A finite lamination with seven pieces and two gaps, and its balloon animal G.

However, at a point of tangency of two circles, ω_∞ will be highly singular with respect to length measure, and it is easy to see that none of the G_i can be a circle in order to satisfy b). In fact, we will see that the boundaries of the G_i are piecewise analytic arcs that make up equal angles at the contact points.

The connected components of $\mathbb{D} \setminus \mathcal{L}$ either have finite hyperbolic area, or meet the circle in one or more non-degenerate intervals. The former are called *gaps* and correspond to points of multiplicity at least three. The latter are in one-to-one correspondence with the G_i, We call them the *pieces* of the lamination and label them P_i. The *balloon animal* of $\mathcal{L}$ and its *balloons* are the domains G and G_i of Proposition 4.1, where we normalize f hydrodynamically ($f(z) = z + O(1/z)$ near ∞).

Proof of Proposition 4.1. The proof of b) and c) is essentially an exercise in conformal welding: Take any smoothly bounded solution to a), form an abstract Riemann surface by glueing discs to ∂G according to the harmonic measure ω_∞, and invoke the uniformization theorem. Because there are singularities at the cut points, we will give a more detailed proof based on the measurable Riemann mapping theorem: First, construct a domain H bounded by C^2-arcs realizing $\mathcal{L}$ and such that near each cut point, ∂H is a star in local coordinates, $\phi(\partial H) = \cup_{k=1}^{2n}[0, e^{\pi i/k}]$: This can easily be done by starting with disjoint actual stars, inductively joining their endpoints by sufficiently smooth curves (for instance hyperbolic geodesics of the complementary domain), and finally correcting the harmonic measures at ∞ of the boundary arcs by an appropriate smooth quasiconformal map as described below. Next, if $h : \Delta \to H$ is a conformal map fixing ∞ and if $h_i : \mathbb{D} \to H_i$ are conformal maps to the bounded complementary components of $\overline{H}$, define homeomorphisms $\phi_i : \mathbb{T} \to \mathbb{T}$ by setting

$$|\phi_i'(x)| = |(h^{-1} \circ h_i)'(x)|/p_{i,j} \quad \text{for} \quad x \in \alpha_{i,j}$$

and normalizing by $\phi_i(1) = 1$, say. Then the ϕ_i are smooth and admit quasiconformal extensions $\Phi_i : \mathbb{D} \to \mathbb{D}$. The Beltrami equation

$$\frac{\overline{\partial} F}{\partial F}(z) = \begin{cases} 0 \quad \text{if} \quad z \in H, \\ \frac{\overline{\partial}}{\partial}\Phi_i \circ h_i^{-1} \quad \text{if} \quad z \in H_i \end{cases}$$

has a quasiconformal solution F, and it is easy to check that $G = F(H)$ satisfies the claim. □

Example 4.2. If $\mathcal{L}$ consists of one chord only, say $(-1, 1)$, then G is the unbounded component of the lemniscate $\{\sqrt{z-1} : |z| = 1\}$ and f is the square root of a quadratic polynomial.

Call a finite lamination $\mathcal{L}$ *well-branched* if for each piece P, $\overline{P} \cap \mathbb{T}$ has one or two connected components. In other words, $\mathcal{L}$ is well-branched if there is no balloon G_i for which ∂G_i contains more than two cut points of $\overline{G}$. The lamination of Figure 4.1 is well-branched. Since we prefer to work with well-branched laminations, we will first show that every finite lamination has a well-branched refinement.

Lemma 4.3. *If $\mathcal{L}'$ is a finite sub-lamination of a maximal lamination $\mathcal{L}$, then there is a finite well-branched lamination $\mathcal{L}''$ with $\mathcal{L}' \subset \mathcal{L}'' \subset \mathcal{L}$.*

This can easily be proved by induction over the number of pieces with more than two cut points: In every such piece, there is a gap subdividing the piece into smaller pieces with fewer cutpoints.

We conclude this section with a simple criterion that guarantees existence of a solution to the realization problem. Let $\mathcal{L}_n$ be an increasing sequence of finite laminations converging to a lamination $\mathcal{L} \supset \cup_n \mathcal{L}_n$ in the sense that for every chord $(a, b) \in \mathcal{L}$ there is a sequence of chords $(a_n, b_n) \in \mathcal{L}_n$ with $a_n \to a$ and $b_n \to b$. Denote $\mathcal{P}_n$ the set of pieces of $\mathcal{L}_n$, and let f_n be a hydrodynamically normalized conformal map of Δ realizing $\mathcal{L}_n$ (that is, $f_n(a) = f_n(b)$ for each $(a, b) \in \mathcal{L}_n$). Denote

$$m_n = \max_{P \in \mathcal{P}_n} \sup_{k \geq n} \operatorname{diam} f_k(\overline{P} \cap \mathbb{T}) \tag{4.3}$$

the largest diameter amongst all images of pieces of generation n. Notice that, by our assumption $\mathcal{L}_n \subset \mathcal{L}_{n+1}$, each of the sets $f_k(\overline{P} \cap \mathbb{T})$ is a finite union of balloons of generation k.

Proposition 4.4. *If $m_n \to 0$ as $n \to \infty$, and if f is any subsequential limit of $(f_n)_{n \geq 1}$ under compact convergence in Δ, then f extends continously to $\overline{\Delta}$, convergence is uniform in Δ, and f realizes $\mathcal{L}$.*

Proof. Each piece of $\mathcal{P}_n$ intersects $\partial\mathbb{D}$ in finitely many arcs. Denote s_n the size of the smallest such arc amongst all pieces of $\mathcal{P}_n$. Notice that $s_n \to 0$ by Beurling's projection theorem, since f_n are normalized and $m_n \to 0$. Then every interval $I \subset \partial\mathbb{D}$ of size $\leq s_n$ is contained in at most two such arcs, hence

$$\operatorname{diam} f_k(I) \leq 2m_n \quad \text{for all} \quad k \geq n.$$

It follows that

$$|f_k(z) - f_k(w)| \leq m'_n \quad \text{for all} \quad k \geq n, z, w \in \Delta, |z - w| < s_n$$

for a sequence $m'_n \to 0$. By pointwise convergence, this also holds for f, so that f is uniformly continuous and extends to $\overline{\Delta}$. It also easily follows that the compact convergence is in fact uniform. Finally, if $(a, b) \in \mathcal{L}$ and $(a_n, b_n) \in \mathcal{L}$ converges to (a, b), then $f(a) = \lim f_n(a_n) = \lim f_n(b_n) = f(b)$ so that f realizes $\mathcal{L}$. □

4.2 The Modulus estimate for finite approximations to $\mathcal{L}$

We inductively construct a sequence $\mathcal{L}_k$ of finite approximations of $\mathcal{L}$ as follows. Set $\mathcal{L}_0 = \emptyset$ and fix $k \geq 1$. For the scale $r = 2^{-k}$ and the dyadic point $x = x_{\ell,k} = \ell/2^k \in \mathbb{T}$ with $1 \leq \ell \leq 2^k$, consider the sets and intervals $A_j = A_j(x, k) \subset I_j = I_j(x, k), A'_j \subset I'_j, 1 \leq j \leq n = n(x, k) \leq N$ of Theorem 1.1. Denote $a_j = a_j(x, k) \in A_j, a'_j \in A'_j$ the point of maximal distance from the point of intersection $x_j \in I_j \cap I'_j$. By the monotonicity of ϕ_j we have $(a_j, a'_{j+1}) \in \mathcal{L}$ for each j. Next, since A_j is uniformly perfect and of size comparable to I_j, it is easy to see that there is a point $b_j \in A_j \cap [a_j, x]$ with $|a_j - b_j| \asymp |b_j - x_j|$ such that $[a_j, b_j] \cap A_j$ is uniformly perfect, with constants only depending on the constant of $\mathcal{L}$. For each j, set $b'_j = \phi(b_j) \in A'_j \cap [x, a'_j]$ so that $(b_j, b'_{j+1}) \in \mathcal{L}$, see Figure 4.2. By the quasisymmetry of ϕ_j we have

$$|a_j - b_j| \asymp |b_j - x_j| \asymp |b'_j - x_j| \asymp |a'_j - b'_j|, \tag{4.4}$$

and $[a'_j, b'_j] \cap A'_j$ is uniformly perfect as well. Now form the set $\hat{\mathcal{L}}_k$ of all such chords (a_j, a'_{j+1}) and (b_j, b'_{j+1}) , namely

$$\hat{\mathcal{L}}_k = \bigcup_{\ell=1}^{2^k} \bigcup_{j=1}^{n(x_\ell,k)} (a_j, a'_{j+1}) \cup (b_j, b'_{j+1}).$$

Applying Lemma 4.3 to the finite sub-lamination $\mathcal{L}' = \mathcal{L}_{k-1} \cup \hat{\mathcal{L}}_k$ of the maximal lamination $\mathcal{L}$, we obtain a well-branched lamination $\mathcal{L}' \subset \mathcal{L}'' \subset \mathcal{L}$ and set

$$\mathcal{L}_k = \mathcal{L}''.$$

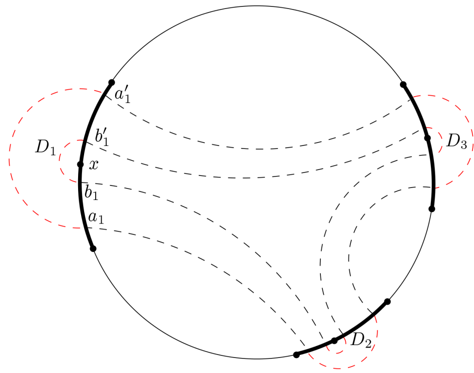


Figure 4.2: The definition of $\mathcal{L}_k$ and the annular neighborhood $\mathcal{A}_k(x)$.

Denoting $D_j = D_{j,k}(x)$ the hyperbolic convex hull (with respect to Δ) of $[a_j, b_j] \cup [a'_j, b'_j]$, we set

$$\mathcal{A}_k(x) = \bigcup_{j=1}^{n(x,k)} D_j, \tag{4.5}$$

see Figure 4.2.

We think of $\mathcal{A}_k(x)$ as an annular neighborhood of x at scale $\asymp 2^{-k}$, and leave the details of the proof of the following Lemma to the reader.

Lemma 4.5. *With*

$$A_{j,k} = A_j \cap [a_j, b_j] \quad \textit{and} \quad A'_{j,k} = A'_j \cap [b'_1, a'_1],$$

the $(D_{j,k}, A_{j,k}, A'_{j,k})$ are conformal $C-$rectangles. Moreover,

$$\mathcal{A}_k(x) \cap \mathcal{A}_{k'}(x) = \emptyset$$

whenever $|k - k'| \geq C'$ and $|x - x'| \leq C'2^{-k}$. Here C and C' only depend on the constant of $\mathcal{L}$.

We now turn to the key modulus estimate. Form the balloon animal $\mathcal{G}_m$ corresponding to $\mathcal{L}_m$. More precisely, apply Proposition 4.1 c) to $\mathcal{L}_m$ with

$$p_{i,j} = 2\omega_\infty(\beta_{i,j})$$

so that

$$\omega(z_i, \beta_{i,1}, G_i) = \omega(z_i, \beta_{i,2}, G_i) = \frac{1}{2} \tag{4.6}$$

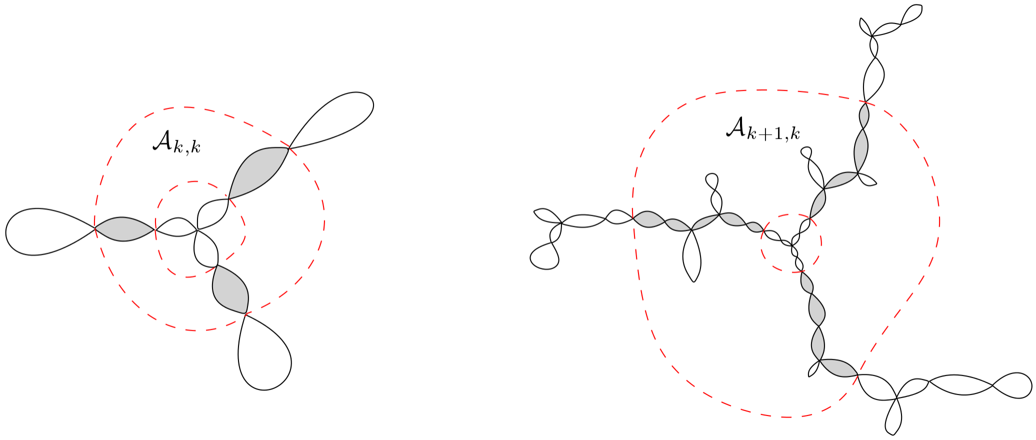


Figure 4.3: Two consecutive balloon animals and annuli $\mathcal{A}_{m,k}(x)$. Shaded are those G_i that allow for a crossing from $f_m(D_j)$ to $f_m(D_j+1)$.

for those i for which the balloon G_i has two boundary arcs (since $\mathcal{L}_m$ is well branched, each G_i either has one or two boundary arcs). Denote $f_m : \Delta \to \mathbb{C} \setminus \mathcal{G}_m$ the corresponding conformal map. Let $k \leq m$, let $x = \ell/2^k \in \mathbb{T}$ be a dyadic point, and consider the image $f_m(\mathcal{A}_k(x))$. The hyperbolic geodesics $f_m(< a_j, a'_j >), 1 \leq j \leq n$ are Jordan arcs whose union forms a Jordan curve surrounding $f_m(x)$, and similarly for the union $\cup_j f_m(< b_j, b'_j >)$. Together these Jordan curves bound a topological annulus

$$\mathcal{A}_{m,k}(x) \supset f_m(\mathcal{A}_k(x)).$$

This annulus can also be obtained from $f_m(\mathcal{A}_k(x))$ by adding those G_i that correspond to the chords of $\mathcal{L}_m$ with endpoints in $\bigcup [a_j, b_j] \cup \bigcup [a'_j, b'_j]$. See Figure 4.3.

Proposition 4.6. *The conformal modulus $M(\mathcal{A}_{m,k}(x))$ is bounded away from zero, with bound depending only on the constant of $\mathcal{L}$.*

Proof. Fix $k \leq m$ and x. Write $\mathcal{A} = \mathcal{A}_{m,k}(x)$ and let Γ be the family of simple closed curves $\gamma \subset \mathcal{A}$ that separate the two boundary components of $\mathcal{A}$. Let $\rho : \mathcal{A} \to [0, \infty]$ be an admissable metric for Γ, that is

$$\int_\gamma \rho |dz| \geq 1 \quad \text{for all} \quad \gamma \in \Gamma.$$

We need to find a lower bound on $\int \rho^2 dxdy$. We assume that

$$\int_{\mathcal{A}} \rho^2 dxdy < \varepsilon,$$

aiming at a contradiction by producing a large family of loops γ in $\mathcal{A}$ of $\rho-$length less than 1 (one such curve suffices for the contradiction). Every loop γ in $\mathcal{A}$ has to cross each

$f_m(D_j), 1 \leq j \leq n$, as well as at least n of the balloons G_i. We will make quantitative the statement that most curves crossing the $f_m(D_j)$ are short, and that many curves crossing the G_i with endpoints corresponding to $\mathcal{L}$ are short as well, so that they can be combined to form the desired loop γ.

Recall the definition (4.5) of $\mathcal{A}_k$ and D_j and set

$$\mathcal{I}_j = [a_j, b_j] \quad \text{and} \quad \mathcal{I}'_j = [a'_j, b'_j]. \tag{4.7}$$

By Lemma 4.5, each $(D_j, \mathcal{I}_j \cap A_j, \mathcal{I}'_j \cap A'_j)$ is a conformal $C-$rectangle. Denote ρ_j the restriction to D_j of pullback of ρ_0 under f_m. For each $j = 1, ..., n$, apply Lemma 3.3 to the image ρ_0 of $\rho_j/\sqrt{\varepsilon}$ under the conformal map ψ_j from D_j onto a rectangle $[0, X_j] \times [0, 1]$,

$$\rho_0(\psi_j(z)|\psi'_j(z)| = \rho(z)/\sqrt{\varepsilon}.$$

Note that ψ_j is a composition of a bilipschitz-map of controlled distortion (namely a composition of a Mobius transformation and a logarithm) and a linear map. We obtain exceptional sets $E_j \subset \mathcal{I}_j$ and $E'_j \subset \mathcal{I}'_j$ such that for every $x \in \mathcal{I}_j \setminus E_j$ and every $y \in \mathcal{I}'_j \setminus E'_j$ there is a curve $\gamma_{x,y;j} \subset D_j$ joining x and y with

$$\int_{\gamma_{x,y;j}} \rho_j |dz| \leq \lambda \sqrt{\varepsilon}.$$

By (3.2) and appropriate choice of the λ, the density of the logarithmic capacity of the exceptional sets in $\mathcal{I}_j$ and $\mathcal{I}'_j$ can be made arbitrarily small,

$$\mathrm{cap}(E_j) \lesssim e^{-\pi\lambda^2} |\mathcal{I}_j| \quad , \quad \mathrm{cap}(E_j) \lesssim e^{-\pi\lambda^2} |\mathcal{I}'_j|.$$

To deal with crossings of the G_i, we will use Proposition 2.12. For each j, the endpoints of the lamination $\mathcal{L}_m$ decompose the arcs $\mathcal{I}_j$ and $\mathcal{I}'_j$ into finitely many open intervals $J_{r,j}$ and $J'_{s,j}$,

$$\mathcal{I}_j \setminus \mathcal{L}_m = \bigcup_r J_{r,j} \ , \quad \mathcal{I}'_j \setminus \mathcal{L}_m = \bigcup_s J'_{s,j} \ .$$

Those $J_{r,j}$ that have non-trivial intersection with the uniformly perfect set A_j are in one-to-one correspondence with the $J'_{s,j+1}$ that intersect A'_{j+1} non-trivially (they correspond to the the two boundary arcs of the shaded G_i in Figure 4.3): If $\mathcal{B}_j = \{r : I_{r,j} \cap A_j \neq \emptyset\}$, then for every $r \in \mathcal{B}_j$ and $x \in I_{r,j} \cap A_j$ there is a unique index s and $y \in I'_{s,j+1} \cap A'_{j+1}$ with $y = \phi_j(x)$. Conversely, with $\mathcal{B}'_j = \{s : I_{s,j} \cap A'_j \neq \emptyset\}$ and $s \in \mathcal{B}'_j$ there is a unique $r \in \mathcal{B}_j$, and we may relabel such that $\mathcal{B}_j = \mathcal{B}'_j$ and I_r corresponds to I'_r. For $j = 1, ..., n$, apply Proposition 2.12 to $U = \psi_j(\mathcal{I}_j \cap A_j), U' = \psi_{j+1}(\mathcal{I}'_{j+1} \cap A'_{j+1})$, $\phi = \psi_{j+1} \circ \phi_j \circ \psi_j^{-1}$, $E = \psi_j(E_j), E' = \psi_{j+1}(E'_{j+1})$ and $J_r = \psi_j(J_{r,j}), J'_r = \psi_{j+1}(J'_{r,j+1})$ for $r \in \mathcal{B}_j$. We obtain an index $r = r_j$ such that

$$\mathrm{cap}(J_{r,j} \cap A_j \setminus (E_j \cup \phi_j^{-1}(E'_{j+1}))) \geq \nu |J_{r,j}|$$

and

$$\operatorname{cap}(J'_{r,j+1} \cap A'_{j+1} \setminus (E'_{j+1} \cup \phi_j(E_j))) \geq \nu |J'_{r,j+1}|.$$

By (4.2) and (4.6), the images of these two sets under the conformal map g_j from G_j onto $\mathbb{D}$, normalized to send the center z_j of G_j to 0, are two sets on different halves of $\partial\mathbb{D}$ of capacity $\geq \nu'$. This is where the special properties of the balloon animals are crucial. By Lemma 3.4 applied to the conformal transport of $\rho/\sqrt{\varepsilon}$ under g_j, if λ is large enough so that $C' e^{-C''\lambda^2} \leq \nu'/2$, then there is a curve $\gamma_j \subset G_j$ with endpoints $f_m(x_j)$ in $f_m(\mathcal{I}_j \setminus E_j)$ and $f_m(y_j)$ in $f_m(\mathcal{I}_{j+1} \setminus E'_{j+1})$ and with $\rho-$length $\leq \lambda\sqrt{\varepsilon}$. The curve

$$\gamma = \bigcup_{j=1}^{n} \gamma_j \cup \bigcup_{j=1}^{n} \gamma_{x_j, y_{j-1}; j} \subset \Gamma$$

has $\rho-$length $\leq 2N\lambda\sqrt{\varepsilon}$ and we obtain a contradiction to the admissability of ρ when ε is small enough. $\square$

4.3 Proof of the "if" part of Theorem 1.1 and of Corollary 1.2

We now have all ingredients to finish the proof of our main result Theorem 1.1.

Proof. Given a maximal non-degenerate lamination $\mathcal{L}$, form the approximations $\mathcal{L}_k$ described in the previous Section 4.2, together with their conformal realizations $f_k : \Delta \to \mathbb{C} \setminus \mathcal{G}_k$ and annuli $\mathcal{A}_{m,k}$ obtained from $f_m(\mathcal{A}_k)$. Denote again $\mathcal{P}_n$ the set of pieces of $\mathcal{L}_n$. Then

$$m_n = \max_{P \in \mathcal{P}_n} \sup_{k \geq n} \operatorname{diam} f_k(\overline{P} \cap \mathbb{T})$$

tends to zero exponentially fast: Indeed, by Lemma 4.5, every piece $P \in \mathcal{P}_n$ is surrounded by n/C' disjoint "annular neighborhoods" of the form $\mathcal{A}_\ell(x)$ (where $x \in \overline{P} \cap \mathbb{T}$ and $\ell = jC', 1 \leq j \leq n/C'$), so that $f_k(\overline{P} \cap \mathbb{T})$ is surrounded by n/C' nested annuli $\mathcal{A}_{k,\ell}(x)$. By Proposition 4.6, all of these annuli have modulus $\geq M_0$, and the claim follows from Lemma 2.6. Let $f = \lim_{j\to\infty} f_{k_j}$ be an arbitrary subsequential limit. By Proposition 4.4, f has a continuous extention to $\overline{\Delta}$ and realizes $\mathcal{L}$, namely $\mathcal{L} = \mathcal{L}_f$.

Notice that the exponential decay of the diameters of the balloons implies the Hölder continuity of f. The stronger John property of $\mathcal{G} = f(\Delta)$ follows from the same modulus estimate, applied to the characterization Theorem 2.4: Indeed, if $A \subset I \subset \mathbb{T}$ are arcs of length $|A| \leq \beta |I|$, then there is a point $x = \ell/2^m \in A$ and a scale $2^m \sim |A|$ such that the annular neighborhood $\mathcal{A}_m(x)$ surrounds A. By Lemma 4.5, there are disjoint nested annular neighborhoods $\mathcal{A}_{m-jC'}(x_j)$. If $\beta \lesssim 2^{-nC'}$, the interval I crosses all $\mathcal{A}_{m-jC'}(x), 1 \leq j \leq n$ (in the sense that at least one of the two intervals $\mathcal{I}_1(x_j, m-jC'), \mathcal{I}'_1(x_j, m-jC')$ defined in (4.7) is contained in I). Consequently, for every k, $f_k(I)$ crosses the annuli $\mathcal{A}_{k,m-jC'}(x_j)$. Since $f_k(A)$ is surrounded by these annuli, by Lemmas 2.6 and 2.7 we have that

$$\log(1+\frac{\operatorname{diam} f(I)}{\operatorname{diam} f(A)}) \geq nM_0 - c$$

and we obtain $\operatorname{diam} f(A) \leq 1/2 \operatorname{diam} f(I)$ if β is sufficiently small. □

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