CONFORMAL METRICS ON THE UNIT BALL IN EUCLIDEAN SPACE

Mario Bonk, Pekka Koskela and Steffen Rohde

Introduction.

In recent years, the theory of conformal and quasiconformal maps of (subdomains of) $\mathbf{R}^n$ has been generalized in many ways. In particular, it has been shown that many properties of these maps are shared by quasiconformal maps between quite general metric spaces, see [**15–17**]. One of the most important points in establishing these results has been the clarification of the basic principles responsible for some of the phenomena in the theory of quasiconformal maps. Our work contributes to the same general program, in that the purpose of this paper is to show that certain features of the theory of conformal mappings on the unit disc $\mathbf{B}^2 = \{z \in \mathbf{C} : |z| < 1\}$ remain valid in a very general setting.

Up to post-composition with an isometry a conformal map $f\colon \mathbf{B}^2 \to \mathbf{C}$ is uniquely determined by $|f'|$. This observation motivates one to think of $|f'|$ as a given "density" ρ on $\mathbf{B}^2$ and to restate mapping properties of f as statements on quantities related to ρ. It is also possible to develop this idea for maps of more general regions $\Omega \subseteq \mathbf{R}^n$. Since we do not want to obscure the main ideas and since certain statements only make sense for the unit ball $\mathbf{B}^n$ in $\mathbf{R}^n$, we restrict ourselves to continuous densities ρ on $\mathbf{B}^n$, $n \geq 2$.

Surprisingly, only two simple conditions on ρ (cf. conditions HI(A) and VG(B) below) are needed to recover a remarkable part of geometric function theory. In addition to $\rho = |f'|$, f conformal, the average a_f (cf. 2.4 below) of the Jacobian of a quasiconformal map $f\colon \mathbf{B}^n \to \mathbf{R}^n$ satisfies these conditions. It can be shown that there exist densities ρ satisfying our conditions that are not related to a map (cf. 2.8 d) below).

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The first condition requires that the density ρ be roughly constant on Whitney type balls. In case $\rho = |f'|$, f conformal, this statement follows from Koebe's distortion theorem.

To state the second condition we associate with ρ a conformal metric d_ρ and a measure μ_ρ in the usual way, cf. Section 1. The condition then says that the measure μ_ρ of balls in the metric d_ρ should not grow faster with the radius than the euclidean volume of balls in $\mathbf{R}^n$. If $\rho = |f'|$, f conformal, then $d_\rho(x, y)$ is the internal euclidean distance of $f(x)$ and $f(y)$ in the image domain $f(\mathbf{B}^2)$, and μ_ρ is the euclidean area of image sets.

As an example of our approach, let us consider the existence of radial limits for conformal mappings f and see how this generalizes to our setting. If $\rho = |f'|$ there seems to be no natural way to recover the euclidean metric of the image domain of f from ρ. Therefore, the mapping properties of a conformal mapping f translate well into statements on ρ only if the internal euclidean metric of the image domain is concerned. Hence the fact that a conformal mapping $f\colon \mathbf{B}^2 \to \mathbf{C}$ has a radial limit along almost every radius has no appropriate interpretation in our setting, since the definition of limits relies on the euclidean metric of the image domain of f. If we rather take the statement that the image of almost every radius has finite length, then this generalizes to the theorem that for a density ρ satisfying the above mentioned conditions, almost every radius has finite length in the metric d_ρ (cf. Theorem 4.4). Similarly, the obvious fact that the Hausdorff dimension of the boundary $\partial f(\mathbf{B}^2)$ is between 1 and 2 has no natural counterpart in our theory. Indeed, there is a canonical way to define the boundary of the metric space $(\mathbf{B}^2, d_\rho)$, and it turns out that this boundary can have infinite Hausdorff dimension (Section 7). However, we show that there is an essential part of the boundary that does have dimension at most 2.

The main tools of our theory are certain modulus estimates (cf. Lemma 3.2) and the Gehring-Hayman theorem (cf. Theorem 3.1); these will be derived from our conditions. Even in the classical cases where the density comes from the Jacobian of a conformal or quasiconformal mapping, our approach leads to new results. For instance, our results about the size of the boundary are stronger than the corresponding classical results from conformal mapping theory, since they rely on the internal metric which is finer than the euclidean metric. The main result of Section 8, an estimate for integral means, was previously only known in the conformal mapping case and was an open problem for quasiconformal maps.

The paper is organized as follows:

Section 1 contains basic notation and definitions. In particular, the two conditions for our densities are introduced.

In Section 2 we discuss examples of densities satisfying our conditions, and give some counterexamples.

In Section 3 we establish our main tools, a modulus estimate and the Gehring-Hayman inequality.

Section 4 contains results corresponding to the boundary behavior of (quasi-) conformal maps. In particular, we prove existence of radial limits, a Lindelöf type theorem and a version of the Beurling projection theorem.

Section 5 is devoted to the study of the density ρ itself. We prove an analog of the Koebe distortion theorem and determine the asymptotic growth of ρ on radii not belonging to small exceptional sets.

In Section 6, some geometric properties of the metric space $(\mathbf{B}^n, d_\rho)$ are discussed and the boundary is introduced.

The Hausdorff dimension of this boundary is studied in Section 7.

In Section 8, the aforementioned generalization of a theorem of Clunie and Pommerenke about integral means is proven.

The appendix Section 9 provides some tools from real analysis needed in Section 7 and Section 8.

1. Basic definitions.

We denote by $|x|$ the euclidean norm of $x \in \mathbf{R}^n$, by $|x-y|$ or $d(x,y)$ the euclidean distance of two points $x, y \in \mathbf{R}^n$, by $\operatorname{diam}(E)$ the diameter of a set $E \subseteq \mathbf{R}^n$, by $\operatorname{dist}(E,F)$ the distance of two set $E, F \subseteq \mathbf{R}^n$, and by $B(a,r)$ for $a \in \mathbf{R}^n$ and $r > 0$ the open ball with center a and radius r. We abbreviate $B(0,1)$ to $\mathbf{B}^n$ and the boundary of $\mathbf{B}^n$ to $\partial\mathbf{B}^n$.

A curve $\gamma\colon I \to \mathbf{R}^n$ is a continuous mapping of an interval $I \subseteq \mathbf{R}$. It is understood that a curve is locally rectifiable. We also denote the image set $\gamma(I)$ of γ by γ. If we speak of a curve in $\mathbf{B}^n$, then we allow the endpoints of the curve to lie in $\partial\mathbf{B}^n$. We denote by $\operatorname{length}(\gamma) \in [0,\infty]$ the euclidean length of γ.

If $\rho\colon \mathbf{B}^n \to [0,\infty]$ is a nonnegative Borel measurable function (a density, for short) on $\mathbf{B}^n$, then the ρ-length of a curve γ in $\mathbf{B}^n$ is

$$\operatorname{length}_\rho(\gamma) = \int_\gamma \rho(x)\,|dx|.$$

Here $|dx|$ denotes integration with respect to euclidean length.

If in addition ρ is continuous and strictly positive, we define

$$d_\rho(x,y) = \inf_\gamma \operatorname{length}_\rho(\gamma) \quad \text{for} \quad x,y \in \mathbf{B}^n,$$

where the infimum is taken over all curves in $\mathbf{B}^n$ with endpoints x and y. Then d_ρ is a metric on $\mathbf{B}^n$ and the topology on $\mathbf{B}^n$ induced by d_ρ is the same as the topology induced by the euclidean metric. Metric notions which refer to the metric d_ρ will have the usual notation valid for the euclidean metric with the additional subscript ρ. For example, $B_\rho(a,r)$ is the open ball with center a and radius r in the metric d_ρ.

Assume $\rho : \mathbf{B}^n \to (0,\infty)$ is a continuous density. We say that ρ satisfies a Harnack type inequality with constant $A \geq 1$ (abbreviated HI(A)) if

$$\mathrm{HI}(A): \quad 1/A \leq \frac{\rho(x)}{\rho(y)} \leq A \text{ whenever } x,y \in B(z, \tfrac{1}{2}(1-|z|)) \text{ for some } z \in \mathbf{B}^n.$$

This condition says that up to the multiplicative constant A the density ρ is constant on each "Whitney ball" $B_z = B(z, \frac{1}{2}(1-|z|))$, $z \in \mathbf{B}^n$. Of course, there is nothing special with the constant $1/2$ in condition HI(A). If we replace it by any other constant $\lambda \in (0,1)$ we get an essentially equivalent condition. More precisely, HI(A) implies that for each $0 < \lambda < 1$ there is a constant $c(A,\lambda) \geq 1$ so that

$$(1.1) \qquad 1/c \leq \frac{\rho(x)}{\rho(y)} \leq c \text{ whenever } x,y \in B(z, \lambda(1-|z|)) \text{ for some } z \in \mathbf{B}^n.$$

Here and in the following the notation $c(A,B,\dots)$ for a constant c shall indicate that the value of c can be chosen in a way which is dependent only on the quantities $A,B,\dots$. If this notation is used the constant c is understood to be positive.

Conversely, if for the density ρ there exist constants $\lambda \in (0,1)$ and $c \geq 1$ so that (1.1) holds, then ρ satisfies HI(A) for some constant $A(\lambda,c)$.

It follows from condition HI(A) for the density ρ that

$$\frac{1}{A}\rho(z)\,\mathrm{length}(\gamma) \le \mathrm{length}_\rho(\gamma) \le A\rho(z)\,\mathrm{length}(\gamma)$$

for every curve γ contained in some B_z, $z \in \mathbf{B}^n$.

With a density ρ we associate a Borel measure μ_ρ on $\mathbf{B}^n$ defined by

$$\mu_\rho(E) = \int_E \rho^n \, dm_n \quad \text{for a Borel set } E \subseteq \mathbf{B}^n.$$

Here integration is with respect to n-dimensional Lebesgue measure m_n. If ρ satisfies HI(A), then

$$\frac{1}{A^n}\rho(z)^n m_n(E) \le \mu_\rho(E) \le A^n \rho(z)^n m_n(E)$$

for every Borel set E contained in some ball B_z, $z \in \mathbf{B}^n$.

We say that ρ satisfies a growth condition for volume with constant $B > 0$ (abbreviated VG(B)) if

$$\mathrm{VG}(B): \qquad \mu_\rho(B_\rho(x,r)) \le Br^n \quad \text{for all} \quad x \in \mathbf{B}^n, r > 0.$$

Apart from Section 2 where we discuss conditions on densities which imply HI and VG we make the following standing assumption.

The density ρ is strictly positive and continuous on $\mathbf{B}^n$ and satisfies HI(A) and VG(B) for some constants $A \ge 1$, $B > 0$.

2. Examples.

We give a list of examples where the conditions of HI(A) and VG(B) are satisfied and a list of non-examples. Mostly, we skip the proofs, since apart from 2.1 none of this material is needed for the rest of the paper.

2.1 Möbius invariance. First we show that the class of densities satisfying the conditions HI and VG remains invariant if we transform the densities with a Möbius transformation.

The hyperbolic metric d_h on $\mathbf{B}^n$ is the metric with the density $h(x) = 1/(1 - |x|^2)$, $x \in \mathbf{B}^n$. The group $\mathrm{Aut}(\mathbf{B}^n)$ of sense preserving diffeomorphisms of $\mathbf{B}^n$ onto itself which preserve the hyperbolic distance of points consists precisely of the sense preserving Möbius transformations T with $T(\mathbf{B}^n) = \mathbf{B}^n$. Each Möbius transformation T is a conformal map.

Suppose $\rho\colon \mathbf{B}^n \to (0,\infty)$ is continuous, $T \in \mathrm{Aut}(\mathbf{B}^n)$, and let J_T be the Jacobian of T. Define $\tilde\rho(x) = \rho(T(x))J_T(x)^{1/n}$ for $x \in \mathbf{B}^n$. Then it follows from the conformality of T that $\mathrm{length}_{\tilde\rho}(\gamma) = \mathrm{length}_\rho(T\circ\gamma)$ for every curve $\gamma \subseteq \mathbf{B}^n$. This implies $d_{\tilde\rho}(x,y) = d_\rho(T(x),T(y))$ for $x,y \in \mathbf{B}^n$. Hence $B_{\tilde\rho}(a,r) = T^{-1}(B_\rho(T(a),r))$ for $a \in \mathbf{B}^n$, $r > 0$. The transformation formula now shows $\mu_{\tilde\rho}(B_{\tilde\rho}(a,r)) = \mu_\rho(B_\rho(T(a),r))$. So if ρ satisfies $\mathrm{VG}(B)$, then $\tilde\rho$ satisfies this condition with the same constant B.

If ρ satisfies $\mathrm{HI}(A)$, then $\tilde\rho$ satisfies $\mathrm{HI}(A')$ with some constant $A'(A)$. In particular, A' does not depend on T. This can be shown by a standard computation using the fact

$$J_T(x)^{1/n} = \frac{1-|T(x)|^2}{1-|x|^2} \quad \text{for} \quad x \in \mathbf{B}^n.$$

This semi-invariance will be important in the proof of Theorem 3.1. In general, we cannot expect $A' = A$. The reason for this is that the condition $\mathrm{HI}(A)$ is not stated in hyperbolically invariant terms. It is easy though to replace $\mathrm{HI}(A)$ by an essentially equivalent condition which has this invariance property. We prefer our original definition, since it can be stated without introducing any additional metrics.

2.2 Conformal mappings.

a) Suppose $f\colon \mathbf{B}^2 \to f(\mathbf{B}^2) = \Omega \subseteq \mathbf{C}$ is a conformal map. Let ρ be the density given by $\rho(x) = |f'(x)|$ for $x \in \mathbf{B}^2$. These densities provide basic examples for densities satisfying $\mathrm{HI}(A)$ and $\mathrm{VG}(B)$. Indeed, for A and B we can take universal constants. As this is one of our prime examples we give a detailed proof:

By the transformation formula the measure μ_ρ corresponds to the measure m_2 in the image domain of f, i.e., $\mu_\rho(E) = m_2(f(E))$ for all Borel sets $E \subseteq \mathbf{B}^2$. The ρ-length of a curve γ in $\mathbf{B}^2$ is equal to the euclidean length of the image of γ under f, i.e., $\mathrm{length}_\rho(\gamma) = \mathrm{length}(f\circ\gamma)$. Accordingly, the metric d_ρ corresponds to the internal euclidean metric in the image domain Ω, i.e.,

$$d_\rho(x,y) = \inf_\gamma \mathrm{length}(\gamma) \quad \text{for} \quad x,y \in \mathbf{B}^2.$$

Here the infimum is taken over all curves γ in Ω with endpoints $f(x)$ and $f(y)$. Therefore, $B_\rho(a,r) \subseteq f^{-1}(B(f(a),r))$ for all $a \in \mathbf{B}^2$ and $r > 0$. It follows that

$$\mu_\rho(B_\rho(a,r)) = m_2(f(B_\rho(a,r))) \le m_2(B(f(a),r)) = \pi r^2.$$

This shows that ρ satisfies VG(π).

It is a well-known consequence of Koebe's distortion theorem for conformal maps that (cf. [**26**], p. 10)

$$\exp(-6d_h(x,y)) \leq \frac{|f'(x)|}{|f'(y)|} \leq \exp(6d_h(x,y)) \quad \text{for all} \quad x,y \in \mathbf{B}^2.$$

Here d_h is the hyperbolic metric on $\mathbf{B}^2$. As $d_h(x,y) \leq 2$ for $x,y \in B_z$, $z \in \mathbf{B}^2$, it follows that ρ satisfies HI(e^{12}).

Hence if ρ is the square root of the Jacobian of a conformal map on $\mathbf{B}^2$, then ρ satisfies HI(A) and GV(B) with universal constants A and B.

b) Suppose $f\colon \mathbf{B}^2 \to \overline{\mathbf{C}}$ is conformal and let $\rho(x) = f^\sharp(x) = |f'(x)|/(1+|f(x)|^2)$ for $x \in \mathbf{B}^2$. Thus ρ is the spherical derivative of f. Then ρ satisfies HI(A) and VG(B) with a constant A only depending on $f^\sharp(0)$ and with $B = \pi$.

2.3 p-valent analytic functions. Suppose $f\colon \mathbf{B}^2 \to \mathbf{C}$ is a p-valent holomorphic function. Let

$$\rho(x) = \frac{1}{m_2(B_x)} \int_{B_x} |f'|\, dm_2 \quad \text{for} \quad x \in \mathbf{B}^2.$$

Then ρ satisfies HI(A) and VG(B) with A and B only depending on p. The reason for this is outlined in 2.4 c) below. Note that we need to take an average of $|f'(x)|$ as the derivative may vanish. If f is p-valent and locally univalent, then we may define $\rho(x) = |f'(x)|$.

2.4 Quasiconformal and quasiregular mappings.

a) Let $f\colon \mathbf{B}^n \to \Omega \subseteq \mathbf{R}^n$ be a K-quasiconformal map and define

$$a_f(x) = \left(\frac{1}{m_n(B_x)} \int_{B_x} J_f\, dm_n\right)^{1/n} \quad \text{for} \quad x \in \mathbf{B}^n.$$

Here J_f is the Jacobian of f. Then $\rho = a_f$ satisfies HI(A) and VG(B) with A and B only depending on K and n. We will briefly outline the reason for this. In this subsection it is convenient to use the notation $A \approx B$ for some quantities A and B if there exists a constant $c(K,n) \geq 1$ such that $(1/c)A \leq B \leq cA$ for all values of the parameters on which A and B may depend. Similarly, $A \lesssim B$ shall indicate that there exists a constant $c(K,n)$ so that $A \leq cB$.

It is known that (cf. [**22**], Lemma 2.5, [**1**])

$$a_f(x) \approx \frac{\operatorname{dist}(f(x), \partial\Omega)}{\operatorname{dist}(x, \partial\mathbf{B}^n)} = \frac{\operatorname{dist}(f(x), \partial\Omega)}{1-|x|} \quad \text{for} \quad x \in \mathbf{B}^n.$$

This and basic distortion estimates for quasiconformal mappings (cf. [**12**], Theorem 3) imply that $\rho = a_f$ satisfies HI(A) with some constant $A(K,n)$.

To show that ρ also satisfies VG(B) with some constant $B(K,n)$ we let $x \in \mathbf{B}^n$ be arbitrary and consider two cases for $r > 0$.

If $r \lesssim a_f(x)(1-|x|)$, then $\mu_\rho(B_\rho(x,r)) \approx a_f(x)^n m_n(B(x, \frac{r}{a_f(x)})) \approx r^n$ by HI(A).

If $r \gtrsim a_f(x)(1-|x|) \approx \operatorname{dist}(f(x), \partial\Omega)$, then there exists a constant $\lambda(K,n)$ so that

$$f\Big(\bigcup_{Q\in\mathcal{W}'} Q\Big) \subseteq B(f(x), \lambda r),$$

where $\mathcal{W}'$ is the set of all cubes Q from the Whitney decomposition of $\mathbf{B}^n$ (cf. Section 7 below) with $Q \cap B_\rho(x,r) \neq \emptyset$. For this use the basic distortion result [**12**, Theorem 3], and Lemma 2.6 in [**22**] that bounds the oscillation of f in terms of the integrals of a_f over appropriate curves. Moreover, by HI(A)

$$\int_Q a_f^n \, dm_n \approx \int_Q J_f \, dm_n$$

for each Whitney cube Q, and hence

$$\mu_\rho(B_\rho(x,r)) \leq \sum_{Q\in\mathcal{W}'} \int_Q J_f \, dm_n \leq m_n(B(f(x), \lambda r)) \approx r^n.$$

This shows that ρ satisfies VG(B) with some constant $B(K,n)$.

b) Let $f\colon \mathbf{B}^n \to \Omega$ be a quasiconformal mapping, where $\Omega \subseteq X$ and X is a linearly locally connected n-regular metric space. Here n-regularity means that X supports a Borel measure ν with $\nu(B(x,r)) \approx r^n$ for each x$\in X$ and and all $0 < r < \operatorname{diam}(X)$. For the precise definitions see [**15–17**]. If we define $\rho(x) = (1-|x|)^{-1}\nu(f(B_x))^{1/n}$, then one can show that ρ satisfies HI(A) and VG(B).

c) If $f\colon \mathbf{B}^n \to \Omega$ is K-quasiregular and p-valent we may either define ρ similarly as in 2.3, or set $\rho = a_f$. By the doubling property of $J_f dm_n$, (cf. [**14**], Theorem 1.5 and Remark 3.6) both definitions coincide up to a bounded factor. Using the doubling property and the fact that $\operatorname{diam} f(B_x) \approx (1-|x|)a_f(x)$ for $x \in \mathbf{B}^n$ ([**14**], 3.7), one can verify conditions HI(A) and VG(B) as in the quasiconformal case above. The constants will depend also on p.

2.5 A_1-weights.

Suppose w is an A_1-weight on $\mathbf{B}^n$. This means that w is a nonnegative locally integrable function on $\mathbf{B}^n$ and that there exists a constant $C > 0$ such that

$$\frac{1}{m_n(B(x,r))}\int_{B(x,r)\cap\mathbf{B}^n} w\,dm_n \le C \operatorname*{essinf}_{y\in B(x,r)\cap\mathbf{B}^n} w(y)$$

for all $x \in \mathbf{B}^n$ and $0 < r < 2$. Here we exclude the trivial case $w = 0$ a.e.

a) If the A_1-weight w is strictly positive, continuous, and satisfies HI(A'), then $\rho = w^{1/n}$ satisfies HI(A) and VG(B) with some constants A and B only dependent on A', C, n.

It is clear that ρ satisfies HI(A) with $A = (A')^{1/n}$. To see that ρ also satisfies VG(B) for some appropriate constant B let $B_\rho(x,r)$ with $x \in \mathbf{B}^n$, $r > 0$, be arbitrary. If we consider the smallest euclidean ball $B(x,R)$ containing $B_\rho(x,r)$, then it follows from the A_1-property of w that $\mu_\rho(B_\rho(x,r)) \le c_1 R^n \inf_{y\in B(x,R)\cap\mathbf{B}^n} w(y)$ for some constant $c_1(C,n)$.

On the other hand, there exists a point $z \in B_\rho(x,r)$ with $|x-z| > R/2$. The point z can be joined with x by a curve in $\mathbf{B}^n$ of ρ-length less than r. For some subcurve γ of this curve we have $\gamma \subseteq \overline{B}(x,R/2)$, $\operatorname{length}(\gamma) \ge R/2$ and $\operatorname{length}_\rho(\gamma) < r$. Decomposing the curve γ into small subcurves which lie in Whitney balls and using HI(A) for ρ and the A_1-property of w it can be shown that $r > \operatorname{length}_\rho(\gamma) \ge c_2 R\big(\inf_{y\in B(x,R)\cap\mathbf{B}^n} w(y)\big)^{1/n}$ for some constant $c_2(A)$. Therefore, $\mu_\rho(B_\rho(x,r)) \le Br^n$ for some constant $B(A,C,n)$.

b) If w is an arbitrary A_1-weight on $\mathbf{B}^n$, define $\tilde{w}(x) = \frac{1}{m_n(B_x)}\int_{B_x} w\,dm_n$ for $x \in \mathbf{B}^n$. It can be shown that $\tilde{w}$ is an A_1-weight fulfilling the hypotheses in a). It follows that $\rho(x) = (\tilde{w}(x))^{1/n}$, $x \in \mathbf{B}^n$, is a density satisfying HI(A) and VG(B).

2.6 Strong A_∞-weights.

Following David and Semmes [7], [**27**] we say that a locally integrable function w on $\mathbf{R}^n$ is a strong A_∞-weight if the measure given by $d\mu(x) = w(x)dm_n$ is a doubling measure and if there is a metric d and a constant C so that

$$C^{-1}d(x,y) \le \mu(B_{x,y})^{1/n} \le C d(x,y)$$

for all x, y, where $B_{x,y}$ is the smallest ball that contains both x and y (it then follows that w is an A_∞-weight). In our setting it is more natural to require this only for points in $\mathbf{B}^n$. For such a weight w we set $\rho(x) = (1-|x|)^{-1}\mu(B_x)^{1/n}$. One can check that ρ satisfies HI(A) and VG(B) with $A = A(n, C)$ and $B = B(n, C)$.

2.7 Lifts.

Let $w \geq 0$ be integrable on $\partial\mathbf{B}^n$ with respect to the spherical measure σ_{n-1}, define sets $S(x) = B(x, 2(1-|x|)) \cap \partial\mathbf{B}^n$ for $x \in \mathbf{B}^n$, and let

$$\rho(x) = \left(\frac{1}{\sigma_{n-1}(S(x))}\int_{S(x)} w\, d\sigma_{n-1}\right)^{1/(n-1)}.$$

Then ρ satisfies HI(A) and VG(B) for some A, B if and only if w is a strong A_∞-weight on $\partial\mathbf{B}^n$.

2.8 Concrete examples.

a) Fix $x_0 \in \partial\mathbf{B}^n$ and $s \in \mathbf{R}$. The density given by $\rho(x) = |x - x_0|^s$ for $x \in \mathbf{B}^n$ satisfies HI(A) and VG(B). The exponents $s \neq -1$ are covered by the quasiconformal mapping $f(x) = (x - x_0)|x - x_0|^s$; for such a quasiconformal mapping we need not use the averaged Jacobian but could work directly with the Jacobian. The remaining case $s = 1$ is handled in the plane by the conformal map $f(x) = \log(x - x_0)$ and in higher dimensions by a quasiconformal substitute of this. Note that with the A_1-theory we could only cover the powers $-1 < s \leq 0$.

b) Let $\alpha \in \mathbf{R}$ and define

$$\rho(x) = \frac{1}{(1-|x|)^\alpha} \quad \text{for} \quad x \in \mathbf{B}^n.$$

Then ρ satisfies HI(A) and VG(B) for some constants A and B if and only if $0 \leq \alpha < 1/n$. For the positive result show that ρ is an A_1-weight. The necessary condition follows from 2.9 below.

c) Let $1/n \leq \alpha < 1$. Then we can find a Cantor set $E \subseteq \partial\mathbf{B}^n$ such that the Hausdorff dimension of E is arbitrarily close to but smaller than $n - n\alpha$ and so that $\rho(x) = d(x, E)^{-\alpha}$ satisfies HI(A) and VG(B).

d) As the results in our paper show, continuous densities $\rho\colon \mathbf{B}^n \to (0, \infty)$ satisfying the conditions HI(A) and VG(B) behave in many respects as the avergages a_f of Jacobians of quasiconformal maps $f\colon \mathbf{B}^n \to \mathbf{R}^n$. It is natural to ask whether all densities ρ arise in this way. More precisely, is it true that for every ρ satisfying our assumptions there exist a constant C and a quasiconformal map $f\colon \mathbf{B}^n \to \mathbf{R}^n$

such that $(1/C)a_f(x) \le \rho(x) \le Ca_f(x)$ for $x \in \mathbf{B}^n$? As the variety of our examples indicates, the answer to this question should be no. It turns out that it is actually not so easy to give an explicit ρ for which one can prove that it is not related to a quasiconformal map in the above sense.

A method to construct explicit examples was pointed out to us by J. Heinonen. The construction is based on the Heisenberg group and results in [**28**]. We will not give the details here, since the argument is quite lengthy.

2.9 Non-examples.

a) If $E \subseteq \partial\mathbf{B}^n$ contains a rectifiable curve, then no density continuous in $\overline{\mathbf{B}^n}$ that vanishes on E satisfies our assumptions. This follows from the Gehring-Hayman inequality (Theorem 3.1 below).

b) If $E \subseteq \partial\mathbf{B}^n$ has positive Hausdorff dimension, then the density ρ cannot satisfy $\int_0^1 \rho(t\zeta)\,|dx| = \infty$ for all $\zeta \in E$. This conclusion follows from the radial limit theorem (Theorem 4.4).

c) If a density grows fast for all radii terminating in a set of positive area, then it cannot satisfy our assumptions. Compare with 2.8 above. Indeed, if ρ satisfies the inequalities HI(A) and VG(A), then for some $\epsilon > 0$

$$\lim_{t\to 1} \rho(t\zeta)(1-t)^{1-1/n+\epsilon} = 0$$

for almost every $\zeta \in \partial\mathbf{B}^n$. This follows from Theorem 5.2 below.

3. The Gehring-Hayman Theorem.

The following inequality plays a central role in our investigation. We use it in combination with modulus estimates to derive some (analogs of) classical results for conformal and quasiconformal mappings. This seems to be a new point of view differing from older treatments of the same matter.

Theorem 3.1. *("Gehring-Hayman Theorem") There is a constant $C(A, B, n)$ with the following property. If γ is a hyperbolic geodesic in $\mathbf{B}^n$ with endpoints in $\overline{\mathbf{B}^n}$ and $\tilde{\gamma}$ is any other curve in $\mathbf{B}^n$ with the same endpoints, then*

$$\operatorname{length}_\rho(\gamma) \le C \operatorname{length}_\rho(\tilde{\gamma}).$$

The hyperbolic geodesic between two points $x, y \in \overline{\mathbf{B}^n}$, $x \ne y$, is the subarc between x and y of the unique circular arc which passes through x and y and is orthogonal to $\partial\mathbf{B}^n$.

Our proof of the Gehring-Hayman theorem is similar in spirit to the earlier arguments, see [**11**], [**18**], [**19**]. For this we need to recall the definition of modulus and establish an important upper bound for the modulus (Lemma 3.2 below).

The modulus $\operatorname{mod}\Gamma \in [0,\infty]$ of a family Γ of curves in $\mathbf{B}^n$ is defined as

$$\operatorname{mod}\Gamma = \inf_{\tilde{\rho}} \int_{\mathbf{B}^n} \tilde{\rho}^n \, dm_n.$$

Here the infimum is taken over all Borel measurable densities $\tilde{\rho}\colon \mathbf{B}^n \to [0,\infty]$ which are admissible. This means $\operatorname{length}_{\tilde{\rho}}(\gamma) \geq 1$ for all $\gamma \in \Gamma$. See for example [**31**] for the basic properties of the modulus.

If E, F are subsets of $\overline{\mathbf{B}^n}$, then we denote by $\operatorname{mod}(E,F;\mathbf{B}^n)$ the modulus of the family of all rectifiable curves in $\mathbf{B}^n$ which connect E and F. By this we mean that the closure of the curve intersects both E and F. If E and F are disjoint continua in $\mathbf{B}^n$, then

$$\phi_n(t) \leq \operatorname{mod}(E,F;\mathbf{B}^n), \tag{3.1}$$

where

$$t = \frac{\operatorname{dist}(E,F)}{\min\{\operatorname{diam}(E), \operatorname{diam}(F)\}}$$

and

$$\phi_n(t) = \frac{\omega_{n-1}}{2\left[\log\left(\lambda_n(1+t)\right)\right]^{n-1}}.$$

Here ω_{n-1} is the surface area of $\partial\mathbf{B}^n$ and $\lambda_n \geq 1$ is a constant that depends only on n. We will also need the fact that $\operatorname{mod}(E,F;\mathbf{B}^n) = \infty$ whenever E, F are connected nondegenerate sets in $\overline{\mathbf{B}^n}$ whose closures have nonempty intersection. For these properties of the modulus see for example [**9**], [**31**].

Lemma 3.2. *There exists a constant $C(B,n) \geq \omega_{n-1}$ with the following property. Let E be a nonempty subset of $\mathbf{B}^n$ and suppose $L \geq \delta > 0$. Assume that $\operatorname{diam}_\rho(E) \leq \delta$ and that Γ is a family of curves in $\mathbf{B}^n$ so that γ has one endpoint in E and $\operatorname{length}_\rho(\gamma) \geq L$ for every $\gamma \in \Gamma$. Then*

$$\operatorname{mod}\Gamma \leq \frac{C}{\left[\log(1 + L/\delta)\right]^{n-1}}.$$

Proof. Define $\tilde{\rho}\colon \mathbf{B}^n \to [0,\infty)$ by

$$\tilde{\rho}(x) = \begin{cases} \dfrac{\rho(x)}{(\log(1+L/\delta))(\delta + \operatorname{dist}_\rho(x,E))} & \text{for} \quad x \in \mathbf{B}^n, \ \operatorname{dist}_\rho(x,E) < L, \\ 0 & \text{elsewhere.} \end{cases}$$

Obviously, the function $\tilde{\rho}$ is Borel measurable and we claim that $\tilde{\rho}$ is an admissible density for the curve family Γ.

To see this let $\gamma \in \Gamma$ be arbitrary. We may assume that $\gamma\colon I \to \mathbf{B}^n$ has an arclength parametrization with $I = [0, \operatorname{length}(\gamma)]$ and $\gamma(0) \in E$. For $s \in I$, let $t(s)$ be the ρ-length of $\gamma|[0,s]$, i.e.,

$$t(s) = \int_0^s \rho(\gamma(u))\, du.$$

We have $\operatorname{dist}_\rho(\gamma(s),E) \le t(s)$ for all $s \in I$. Since $t(s)$ is a continuous increasing function of s and $t(\operatorname{length}(\gamma)) = \operatorname{length}_\rho(\gamma) \ge L$, there exists a smallest number $s_0 \in I$ with $t(s_0) = L$. It now follows from the definition of $\tilde{\rho}$ that

$$\begin{aligned} \operatorname{length}_{\tilde{\rho}}(\gamma) &\ge \frac{1}{\log(1+L/\delta)} \int_0^{s_0} \frac{\rho(\gamma(s))}{\delta + \operatorname{dist}_\rho(\gamma(s),E)}\, ds \\ &\ge \frac{1}{\log(1+L/\delta)} \int_0^{L} \frac{dt}{\delta + t} = 1. \end{aligned}$$

Hence $\tilde{\rho}$ is admissible.

Select a point $x_0 \in E$ and let $k \in \mathbf{N}$ be the smallest number with $\delta + L \le 2^{k+1}\delta$. Then it follows from $1 \le L/\delta$ and the definition of k that

$$k \le \frac{\log(1+L/\delta)}{\log 2}. \tag{3.2}$$

Since $\operatorname{diam}_\rho(E) \le \delta$ we have $\{x \in \mathbf{B}^n : \operatorname{dist}_\rho(x,E) < L\} \subseteq B_\rho(x_0, 2^{k+1}\delta)$.

Let $B_j = B_\rho(x_0, 2^j\delta)$ for $j \in \mathbf{N}$. Then $\tilde{\rho}$ vanishes outside B_{k+1} and so

$$\begin{aligned} \operatorname{mod}\Gamma &\le \int_{\mathbf{B}^n} \tilde{\rho}^n\, dm_n = \int_{B_{k+1}} \tilde{\rho}^n\, dm_n \\ &= \int_{B_1} \tilde{\rho}^n\, dm_n + \sum_{j=1}^{k} \int_{B_{j+1}\setminus B_j} \tilde{\rho}^n\, dm_n. \end{aligned}$$

Since $\operatorname{dist}_\rho(x,E) \geq 2^j\delta - \delta$ for $x \in B_{j+1}\backslash B_j$ and $j \in \mathbf{N}$, we get

$$\begin{aligned}
\operatorname{mod}\Gamma &\leq \frac{1}{[\log(1+L/\delta)]^n}\left[\int_{B_1}\frac{\rho^n}{\delta^n}\,dm_n + \sum_{j=1}^{k}\int_{B_{j+1}\backslash B_j}\frac{\rho^n}{2^{jn}\delta^n}\,dm_n\right] \\
&\leq \frac{1}{[\delta\log(1+L/\delta)]^n}\left[\mu_\rho(B_1) + \sum_{j=1}^{k}\frac{1}{2^{jn}}\mu_\rho(B_{j+1})\right].
\end{aligned}$$

Condition $\mathrm{VG}(B)$ and (3.2) imply

$$\operatorname{mod}\Gamma \leq \frac{2^n(k+1)B}{[\log(1+L/\delta)]^n} \leq \frac{2^{n+1}B}{\log 2[\log(1+L/\delta)]^{n-1}}.$$

The lemma follows. $\square$

Proof of Theorem 3.1. Suppose that the endpoints o γ and $\tilde{\gamma}$ are $a, b \in \overline{\mathbf{B}^n}$. There exists a Möbius transformation $T \in \operatorname{Aut}(\mathbf{B}^n)$ so that $\gamma' = T^{-1}(\gamma)$ is a line segment containing the origin. If at least one of the endpoints of γ is contained $\mathbf{B}^n$, then T may in addition be chosen so that one of the endpoints of γ' is the origin. In any case γ' is a hyperbolic geodesic with the same endpoints as the curve $\tilde{\gamma}' = T^{-1}(\gamma)$. If we replace our original density ρ by $\rho' = (\rho\circ T)(J_T)^{1/n}$, then ρ' satisfies $\mathrm{HI}(A')$ and $\mathrm{VG}(B)$ with A' depending on A only (and not on T, see 2.1). Since $\operatorname{length}_{\rho'}(\gamma') = \operatorname{length}_\rho(\gamma)$ and $\operatorname{length}_{\rho'}(\tilde{\gamma}') = \operatorname{length}_\rho(\tilde{\gamma})$, the statement of the theorem is reduced to the case where the hyperbolic geodesic contains the origin and the origin is an endpoint of γ if at least one of the endpoints is contained in $\mathbf{B}^n$. Therefore, we have to consider the following cases for a and b: $a \in \partial\mathbf{B}^n$ and $b = -a$ or $a \in \overline{\mathbf{B}^n}$ and $b = 0$. For the rest of the proof, we will assume $a \in \partial\mathbf{B}^n$ and $b = 0$. In the other cases minor modifications of the argument are necessary which we leave to the reader.

We decompose γ into subcurves and assign to each of these subcurves a subcurve of $\tilde{\gamma}$ whose ρ-length controls the ρ-length of the corresponding subcurve of γ by a factor $C(A,B,n)$. The subcurves of $\tilde{\gamma}$ will be chosen in a way so that they do not essentially overlap and the sum of their ρ-lengths is less or equal to the ρ-length of $\tilde{\gamma}$. The statement of the theorem then follows.

For the construction of these subcurves we consider annuli $A_j = \overline{B}(a, 2^{-j}) \setminus B(a, 2^{-j-1})$ for $j \in \mathbf{N}_0$. For each j, let $\gamma_j = \gamma \cap A_j$. The curve γ_j is the closed line segment $[2^{-j}a, \frac{1}{2}2^{-j}a]$ with center $a_j = \frac{3}{4}2^{-j}a$. Select a closed subcurve $\tilde{\gamma}_j$ of $\tilde{\gamma}$ that joins the boundary components of A_j in $A_j \cap \mathbf{B}^n$. The curves γ_j, $j \in \mathbf{N}_0$, form a decomposition of $\gamma = [0, a)$ and the curves $\tilde{\gamma}_j$, $j \in \mathbf{N}_0$, are essentially non overlapping subcurves of $\tilde{\gamma}$.

For $j \in \mathbf{N}_0$ we have $\operatorname{dist}(\gamma_j, \tilde{\gamma}_j) \leq 2^{-j} \leq 2\min\{\operatorname{diam}(\gamma_j), \operatorname{diam}(\tilde{\gamma}_j)\}$. Hence the basic modulus estimate (3.1) gives $\operatorname{mod}(\gamma_j, \tilde{\gamma}_j; \mathbf{B}^n) \geq c_1(n) > 0$.

If $c \geq 1$ and $\operatorname{length}_\rho(\alpha) \geq c \operatorname{length}_\rho(\tilde{\gamma}_j)$ for all curves α that join γ_j and $\tilde{\gamma}_j$ in $\mathbf{B}^n$, then Lemma 3.2 shows that

$$\operatorname{mod}(\gamma_j, \tilde{\gamma}_j; \mathbf{B}^n) \leq \frac{c_2}{[\log(1+c)]^{n-1}}$$

with some constant $c_2(B, n)$. This is impossible if c exceeds a number depending on c_1, c_2, and n. It follows that γ_j and $\tilde{\gamma}_j$ can be joined by a curve α_j in $\mathbf{B}^n$ with $\operatorname{length}_\rho(\alpha_j) \leq c_3 \operatorname{length}_\rho(\tilde{\gamma}_j)$, where c_3 depends only on B and n.

We now consider two cases according to whether $\operatorname{length}(\alpha_j)$ is or is not substantially smaller than $\operatorname{length}(\gamma_j)$.

Suppose $\operatorname{length}(\alpha_j) \leq \frac{1}{10} \operatorname{length}(\gamma_j)$, say. Since α_j connects γ_j and $\tilde{\gamma}_j$, we have $\operatorname{dist}(\gamma_j, \tilde{\gamma}_j) \leq \frac{1}{10} \operatorname{length}(\gamma_j)$. On the other hand, $\operatorname{length}(\gamma_j) \leq \operatorname{length}(\tilde{\gamma}_j)$. It follows that $\tilde{\gamma}_j$ must have a subcurve of euclidean length comparable to $\operatorname{length}(\gamma_j)$ near the Whitney ball B_{a_j} containing γ_j. It then follows from HI(A) that $\operatorname{length}_\rho(\gamma_j) \leq c_4 \operatorname{length}_\rho(\tilde{\gamma}_j)$ for some $c_4(A)$.

If $\frac{1}{10} \operatorname{length}(\gamma_j) < \operatorname{length}(\alpha_j)$, then the same argument using HI(A) implies $\operatorname{length}_\rho(\gamma_j) \leq c_5 \operatorname{length}_\rho(\alpha_j)$ for some $c_5(A)$. Hence $\operatorname{length}_\rho(\gamma_j) \leq c_6 \operatorname{length}_\rho(\tilde{\gamma}_j)$ with $c_6(A, B, n) = c_3 c_5$.

So if $C(A, B, n) = \max\{c_4, c_6\}$, then $\operatorname{length}_\rho(\gamma_j) \leq C \operatorname{length}_\rho(\tilde{\gamma}_j)$. The desired estimate follows by summing over j. $\square$

Remark 3.3. If $\rho = |f'|$ for some conformal map f of $\mathbf{B}^2$, then we recover the classical Gehring-Hayman theorem. But if ρ is associated to a quasiconformal map as in 2.4, then the integrability of ρ over γ does not imply the rectifiability of $f(\gamma)$. In this case, $\operatorname{length}_\rho(\gamma)$ is comparable to the "discrete length" of $f(\gamma)$ as defined in [**19**].

4. Boundary behavior.

The results of this section are analogs of classical statements about boundary behavior, cf. [**26**]. We show that most of the radii have finite ρ-length, see Theorem 4.4 below. This corresponds to the classical theorem of Beurling about existence of radial limits, [**4**]. We begin by establishing an estimate on euclidean distances in terms of our new metric (this corresponds to the Beurling projection theorem).

Lemma 4.1. *There exist constants $\alpha(B,n) \in (0,1)$ and $C'(A,B,n) > 0$ so that*

$$\operatorname{diam}(E) \le C' \left(\frac{\operatorname{diam}_\rho(E)}{\rho(0)} \right)^\alpha$$

for each continuum $E \subseteq \mathbf{B}^n$. In particular

$$|x-y| \le C' \left(\frac{d_\rho(x,y)}{\rho(0)} \right)^\alpha \quad \textit{for} \quad x,y \in \mathbf{B}^n.$$

In fact we can take $\alpha = \left(\frac{\omega_{n-1}}{2C}\right)^{1/(n-1)}$, where C is the constant from Lemma 3.2.

Note that if $\rho = |f'|$ for a conformal map f of the unit disc, an inequality similar to that of Lemma 3.2 holds with $C = 2\pi$ and the proof below gives $\alpha = 1/2$ in this case. Thus we recover the classical sharp bound. In the quasiconformal setting such an argument also produces essentially sharp results (cf. [**22**]).

Proof. Let α be as above. Fix a small ball centered at the origin, $B(0, \frac{1}{10})$ say.

If $\operatorname{diam}_\rho(E)$ is comparable to or bigger than $\rho(0)$, then an inequality of the required type is true as $\operatorname{diam}(E) \le 2$.

So we may assume $\operatorname{diam}_\rho(E) \le \frac{\rho(0)}{10A}$. If E intersects $B(0, \frac{1}{10})$, the continuum E will be contained in a ball $B(0,r)$ for some fixed $r \in (0,1)$ not depending on E. From HI(A) it then follows that $\rho(0)\operatorname{diam}(E)$ and $\operatorname{diam}_\rho(E)$ are comparable. The desired inequality follows also in this case as $0 < \alpha < 1$.

So we are reduced to the main case that $\operatorname{diam}_\rho(E) \le \frac{\rho(0)}{10A}$ and $E \cap B(0, \frac{1}{10}) = \emptyset$. For each curve γ that joins $\overline{B}(0, \frac{1}{20})$ and E we have

$$\operatorname{length}_\rho(\gamma) \ge \frac{\rho(0)}{20A} \ge \operatorname{diam}_\rho(E) > 0.$$

We deduce from Lemma 3.2 that

$$\operatorname{mod}(\overline{B}(0,\tfrac{1}{20}),E;\mathbf{B}^n) \le \frac{C}{[\log(1+\rho(0)/(20A\operatorname{diam}_\rho(E)))]^{n-1}}.$$

On the other hand, we have $\operatorname{diam}(E) \le 2$ and so

$$t = \frac{\operatorname{dist}(\overline{B}(0,\frac{1}{20}),E)}{\min\{\operatorname{diam}(\overline{B}(0,\frac{1}{20})),\operatorname{diam}(E)\}} \le \frac{20}{\operatorname{diam}(E)}.$$

Thus, from (3.1) we get

$$\operatorname{mod}(\overline{B}(0,\tfrac{1}{20}),E;\mathbf{B}^n) \ge \frac{\omega_{n-1}/2}{[\log(\lambda_n(1+20/\operatorname{diam}(E))]^{n-1}}.$$

The desired inequality follows from these modulus estimates by calculation. □

The next two statements can be considered as a version of the Lindelöf theorem (cf. [**24**], [**10**]): If a conformal map on $\mathbf{B}^2$ has a limit along a curve ending at a boundary point $\zeta \in \partial\mathbf{B}^2$, then f has an angular limit at ζ.

Theorem 4.2. *If (x_k) is a d_ρ-Cauchy sequence in $\mathbf{B}^n$ and $(x_k) \to \zeta \in \partial\mathbf{B}^n$ (in the euclidean sense), then $\operatorname{length}_\rho([0,\zeta)) < \infty$. Moreover, if (y_k) is any sequence on the radius $[0,\zeta)$ with $(y_k) \to \zeta$, then $\lim_{k\to\infty} d_\rho(x_k,y_k) = 0$.*

Proof. We can inductively construct a subsequence (z_k) of (x_k) with $d_\rho(z_k,z_{k+1}) \le \frac{1}{2^k}$ for $k \in \mathbf{N}$. By definition of the metric d_ρ, for each $k \in \mathbf{N}$ there exists a curve β_k in $\mathbf{B}^n$ starting in z_k and ending at z_{k+1} so that

$$\operatorname{length}_\rho(\beta_k) \le 2d_\rho(z_k,z_{k+1}) \le \frac{1}{2^{k-1}}.$$

Furthermore, let β_0 be any curve in $\mathbf{B}^n$ which has finite ρ-length and connects 0 and z_1.

The curves β_k can be joined to give a curve $\tilde{\gamma}$. More precisely, there exists a curve $\tilde{\gamma} : [0,1) \to \mathbf{B}^n$ with $\tilde{\gamma}(0) = 0$, $\tilde{\gamma} = \beta_0 \cup \beta_1 \cup \dots$ and

$$\operatorname{length}_\rho(\tilde{\gamma}) = \sum_{j=0}^{\infty} \operatorname{length}_\rho(\beta_j) < \infty.$$

Since $z_k \in \beta_k \subseteq \tilde{\gamma}$ for $k \in \mathbf{N}$ there exists $t_k \in [0,1)$ with $z_k = \gamma(t_k)$. We have $(t_k) \to 1$.

For $t \in [0,1)$ define $F_t = \tilde{\gamma}([t,1))$. By Lemma 4.1 there exist constants c, $\alpha > 0$ not depending on t so that

$$\operatorname{diam}(F_t) \leq c(\operatorname{diam}_\rho(F_t))^\alpha \leq c(\operatorname{length}_\rho(F_t))^\alpha \quad \text{for} \quad t \in [0,1).$$

Since $\tilde{\gamma}$ has finite ρ-length we have $\lim_{t\to 1} \operatorname{length}_\rho(F_t) = 0$ and so $\lim_{t\to 1} \operatorname{diam}(F_t) = 0$. Therefore, $\lim_{t\to 1} \tilde{\gamma}(t)$ exists in the euclidean sense and

$$\lim_{t\to 1} \tilde{\gamma}(t) = \lim_{k\to\infty} z_k = \zeta.$$

Defining $\tilde{\gamma}(1) = \zeta$, we get a continuous extension of γ to $[0,1]$.

The Gehring-Hayman theorem 3.1 shows that the hyperbolic geodesic γ joining 0 and ζ, i.e. the radius $[0,\zeta)$, has finite ρ-length since this is true for the curve $\tilde{\gamma}$ with the same endpoints.

For $t \in [0,1)$ let $E_t = [t\zeta, \zeta)$. The fact that $[0,\zeta)$ has finite ρ-length implies $\lim_{t\to 1} \operatorname{diam}_\rho(E_t) = 0$.

If Γ_t is the family of curves joining E_t and F_t in $\mathbf{B}^n$, then $\operatorname{mod}\Gamma_t = \infty$, since $\operatorname{diam}(E_t)$, $\operatorname{diam}(F_t) > 0$ and $\zeta \in \overline{E_t} \cap \overline{F_t}$. This implies that there exists a curve $\alpha_t \in \Gamma_t$ with $\operatorname{length}_\rho(\alpha_t) < \operatorname{diam}_\rho(E_t)$. For otherwise, $\operatorname{mod}\Gamma_t < \infty$ by Lemma 3.2. Therefore,

$$\operatorname{diam}_\rho(E_t \cup F_t) \leq \operatorname{length}_\rho(\alpha_t) + \operatorname{diam}_\rho(E_t) + \operatorname{diam}_\rho(F_t) \leq 2\operatorname{diam}_\rho(E_t) + \operatorname{diam}_\rho(F_t)$$

and so $\lim_{t\to 1} \operatorname{diam}_\rho(E_t \cup F_t) = 0$.

Since for every $t \in [0,1)$ the sets E_t and F_t contain tails of the sequences (y_k) and (z_k), respectively, we get $\lim_{k\to\infty} d_\rho(z_k, y_k) = 0$. Now $\lim_{k\to\infty} d_\rho(x_k, y_k) = 0$ follows from the fact that (z_k) is a subsequence of the ρ-Cauchy sequence (x_k). $\square$

For $\zeta \in \partial\mathbf{B}^n$, $\lambda \in (0,1)$ and $\epsilon \in (0,1]$ let

$$\operatorname{Cone}(\zeta, \lambda, \epsilon) = \bigcup \{B(t\zeta, \lambda(1-t)) : 1 - \epsilon \leq t < 1\}$$

be a Stolz cone at ζ. It follows from $\mathrm{HI}(A)$ that the d_ρ-diameter of $\operatorname{Cone}(\zeta,\lambda,\epsilon)$ is comparable to the d_ρ-diameter of $[(1-\epsilon)\zeta, \zeta)$. More precisely, given $\lambda \in (0,1)$ there exists a constant $C(A,\lambda)$ such that

$$\operatorname{diam}_\rho(\operatorname{Cone}(\zeta,\lambda,\epsilon)) \leq C \operatorname{diam}_\rho([(1-\epsilon)\zeta,\zeta)) \quad \text{for} \quad \zeta \in \partial\mathbf{B}^n,\ \epsilon \in (0,1]. \tag{4.1}$$

Theorem 4.3. *Suppose* $\zeta \in \partial\mathbf{B}^n$ *and* $\text{length}_\rho([0,\zeta)) < \infty$*. Then for all* $\lambda \in (0,1)$

$$\lim_{\epsilon\to 0} \text{diam}_\rho(\text{Cone}(\zeta,\lambda,\epsilon)) = 0.$$

Proof. If $\text{length}_\rho([0,\zeta)) < \infty$, then $\lim_{\epsilon\to 0} \text{diam}_\rho([(1-\epsilon)\zeta,\zeta)) = 0$. The statement now follows from (4.1). □

To state the next theorem we recall the notion of sets E with vanishing n-capacity $\text{cap}_n(E)$. If $C^\infty(\mathbf{R}^n)$ is the set of indefinitely differentiable real-valued functions on $\mathbf{R}^n$ and $K \subseteq \overline{\mathbf{B}^n}$ is compact, then $\text{cap}_n(K) = 0$ if and only if

$$\inf\left\{\int_{\mathbf{R}^n} |\nabla u|^n \, dm_n : u \in C^\infty(\mathbf{R}^n),\ u|K \geq 1 \text{ and } u|(\mathbf{R}^n \setminus B(0,2)) \equiv 0\right\} = 0.$$

If $E \subseteq \overline{\mathbf{B}^n}$ is any Borel set, then $\text{cap}_n(E) = 0$ if and only if $\text{cap}_n(K) = 0$ for all compact sets $K \subseteq E$.

For our present purpose we need the following fact. If $F \subseteq \mathbf{B}^n$ is any continuum and $\Gamma(E,F)$ is the family of curves in $\mathbf{B}^n$ connecting F and a Borel set $E \subseteq \partial\mathbf{B}^n$, then $\text{mod}\,\Gamma(E,F) = 0$ implies $\text{cap}_n(E) = 0$. Conversely, if $\text{cap}_n(E) = 0$, then $\text{mod}\ \Gamma(E,F) = 0$ for all continua $F \subseteq \mathbf{B}^n$.

Recall that sets of n-capacity zero have Hausdorff dimension zero.

Theorem 4.4. *("Radial Limit Theorem") There is a Borel set* $E \subseteq \partial\mathbf{B}^n$ *with* $\text{cap}_n(E) = 0$ *so that*

$$\text{length}_\rho([0,\zeta)) < \infty \quad \textit{for all} \quad \zeta \in \partial\mathbf{B}^n \setminus E.$$

Proof. For $k \in \mathbf{N}$ let $E_k = \{\zeta \in \partial\mathbf{B}^n : \text{length}_\rho([0,\zeta)) \leq k\}$. To see that these sets are closed let (ζ_i) be a sequence in some E_k with $(\zeta_i) \to \zeta$. If $\lambda \in [0,1)$ is arbitrary

$$\text{length}_\rho([0,\lambda\zeta]) = \int_0^\lambda \rho(t\zeta)\,dt = \lim_{i\to\infty} \int_0^\lambda \rho(t\zeta_i)\,dt \leq k.$$

This implies

$$\text{length}_\rho([0,\zeta)) = \lim_{\lambda\to 1} \text{length}_\rho([0,\lambda\zeta)) \leq k$$

and so $\zeta \in E_k$.

Let $E = \{\zeta \in \partial\mathbf{B}^n : \text{length}_\rho([0,\zeta)) = \infty\}$. Since $E = \partial\mathbf{B}^n \backslash \left(\bigcup_{k\in\mathbf{N}} E_k\right)$ and the sets E_k are closed, the set E is a Borel set.

We claim $\text{cap}_n(E) = 0$. Assume this is not the case. Let $F = \overline{B}(0, 1/2)$ and consider the family Γ of curves in $\mathbf{B}^n$ connecting E and F. By the above remark $\text{cap}_n(E) > 0$ implies mod $\Gamma > 0$. This in turn implies that there exists a curve $\alpha \in \Gamma$ with $\text{length}_\rho(\alpha) < \infty$. For if this were not the case, then mod $\Gamma = 0$ by Lemma 3.2.

Let $\zeta \in E$ and $x \in F$ be the endpoints of α. Since x can be connected to 0 by a curve of finite ρ-length, there exists a curve $\tilde{\gamma}$ in $\mathbf{B}^n$ with endpoints 0 and ζ and $\text{length}_\rho(\tilde{\gamma}) < \infty$. The Gehring-Hayman theorem implies that $\text{length}_\rho([0,\zeta)) < \infty$. This is a contradiction to $\zeta \in E$ by definition of E. □

Remark 4.5. a) The reader familiar with conformal mappings may find the following observation amusing. Suppose that $\text{diam}_\rho(\mathbf{B}^n) < \infty$. Then $\text{length}_\rho([0,\zeta)) \leq M < \infty$ for *all* $\zeta \in \partial\mathbf{B}^n$. This follows from the Gehring-Hayman theorem.

b) For conformal maps of the unit disc, Theorem 4.4 implies the (well-known) rectifiabilty of the image of capacity-a.e. radius. For quasiconformal maps, the integrability of $\rho = a_f$ does not imply the rectifiability of the image. In fact, there can be a set of Hausdorff dimension $n-1$ so that the image of no radius ending in this set is rectifiable, [**19**]. However, it easy to see that integrability of $\rho = a_f$ on a radius implies the existence of the radial limit and we recover the existence of radial limits capacity-a.e., cf. [**33**].

5. Asymptotic behavior of ρ.

We begin by stating pointwise estimates for $\rho(x)$. The Harnack type inequality HI(A) immediately gives estimates for $\rho(x)$ in terms of $(1-|x|)^{c(A)}$. However, these exponents do not reduce to the classical ones in the case of conformal maps. Our first result gives such exponents in terms of n and the constant B from the volume growth condition VG(B). Our second result gives a better growth estimate for most of the radii, cf. [**21**] for the quasiconformal case.

Theorem 5.1. *There exist constants $\beta(B,n) > 1$, $C_1(A,B,n)$, $C_2(A,B,n)$ so that*

$$C_1(1-|x|)^{\beta-1} \leq \frac{\rho(x)}{\rho(0)} \leq C_2 \frac{1}{(1-|x|)^{\beta+1}} \quad \textit{for} \quad x \in \mathbf{B}^n.$$

In fact, we may take $\beta = \left(\frac{2C}{\omega_{n-1}}\right)^{1/(n-1)}$, where C is the constant from Lemma 3.2. In the case of conformal maps in the plane, the claim of Lemma 3.2 holds with $C = 2\pi$ and we recover the right exponent in the classical result.

Proof. The lower bound is a consequence of Lemma 4.1. Indeed, by letting $E = B_x$ we have $\operatorname{diam}(E) = 1 - |x|$ and $\operatorname{diam}_\rho(E) \le A\rho(x)(1-|x|)$. The desired estimate follows.

To prove the upper bound let $x \in \mathbf{B}^n$ be arbitrary. We may assume that x is close to $\partial\mathbf{B}^n$ and $\rho(x)(1-|x|)$ is comparable to $\rho(0)$. For otherwise, an inequality of the required type is trivial by HI(A). The proof now continues similarly as the proof of Lemma 4.1.

Consider the two continua $E = \overline{B_0}$ and $F = \overline{B_x}$ and the family Γ of curves in $\mathbf{B}^n$ connecting E and F. If x is sufficiently close to $\partial\mathbf{B}^n$, every curve $\gamma \in \Gamma$ has ρ-length greater than a quantity comparable to $\rho(x)(1-|x|)$. Combining the modulus estimate in (3.1) with Lemma 3.2 the statement follows. We leave the details to the reader. □

For the second result of this section we need to define the meaning of sets of α-Hausdorff measure zero. Let $\alpha > 0$ and $E \subseteq \mathbf{R}^n$. We write $H^\alpha(E) = 0$ provided for each $\epsilon > 0$ there is a countable covering $E \subseteq \bigcup_{k\in\mathbf{N}} B(x_k, r_k)$ with $\sum_{k=1}^\infty r_k^\alpha < \epsilon$.

Theorem 5.2. *Let $0 < \alpha \le n-1$. Then there exists a set $E \subseteq \partial\mathbf{B}^n$ with $H^\alpha(E) = 0$ such that for all $\zeta \in \partial\mathbf{B}^n \setminus E$*

$$\rho(t\zeta) = o((1-t)^{-1+\alpha/n}) \quad \text{as} \quad t \to 1.$$

Proof. Define $E_\infty = \{\zeta \in \partial\mathbf{B}^n : \operatorname{length}_\rho([0,\zeta)) = \infty\}$. Then $\operatorname{cap}_n(E_\infty) = 0$ by Theorem 4.4. This implies $H^\alpha(E_\infty) = 0$.

Moreover, for $j \in \mathbf{N}$ define $G_j = \{\zeta \in \partial\mathbf{B}^n : \operatorname{length}_\rho([0,\zeta)) \le j\}$ and

$$F_j = \bigcup_{\zeta \in G_j} \operatorname{Cone}(\zeta, 1/2, 1).$$

Then F_j is open and by (4.1) we have $\operatorname{diam}_\rho(F_j) < \infty$. In particular, $\mu_\rho(F_j) < \infty$ by VG(B).

If we define $u_j(x) = \rho(x)^n$ for $x \in F_j$ and $u_j(x) = 0$ elsewhere, then $u_j \in L^1(\mathbf{B}^n)$. Hence there exists a set $E_j \subseteq \partial\mathbf{B}^n$ with $H^\alpha(E_j) = 0$ such that for all $\zeta \in \partial\mathbf{B}^n \setminus E_j$

$$\int_{B(\zeta,r)\cap\mathbf{B}^n} u_j\,dm_n = o(r^\alpha) \quad \text{as} \quad r \to 0,$$

cf. [**32**], p. 118. In particular, for each $\zeta \in G_j \setminus E_j$

$$\int_{B_{t\zeta}} \rho^n\,dm_n = \int_{B_{t\zeta}} u_j\,dm_n \le \int_{B(\zeta,\frac{3}{2}(1-t))} u_j\,dm_n = o((1-t)^\alpha) \quad \text{as} \quad t \to 1,$$

and so

$$\rho(t\zeta)^n \le 2^n A^n (1-t)^{-n} \int_{B_{t\zeta}} \rho^n\,dm_n = o((1-t)^{\alpha-n}) \quad \text{as} \quad t \to 1.$$

Since $\bigcup_{j\in\mathbf{N}} G_j = \partial\mathbf{B}^n \setminus E_\infty$, the desired estimate holds for all $\zeta \in \partial\mathbf{B}^n \setminus E$, where $E = E_\infty \cup \bigcup_{j\in\mathbf{N}} E_j$. Since E is a countable union of sets of α-Hausdorff measure 0, we have $H^\alpha(E) = 0$ and the lemma follows. □

Remark 5.3. The examples in Section 2 show that the exponent $-1+\alpha/n$ in Theorem 5.2 is sharp for the entire class of densities ρ. However, for a fixed density ρ a better estimate holds at least when $\mu_\rho(\mathbf{B}^n) < \infty$: in this case the claim of Theorem 5.2 holds with $-1+\alpha/n$ replaced by $-1+\alpha/n+\epsilon$, where $\epsilon = \epsilon(n, B)$. To see this one can use Theorem 8.1 below: the growth order of the integral means of ρ^{n-1} is no worse than $(1-t)^{(1-n)/n+2\epsilon}$, where $\epsilon > 0$ depends only on n. It follows that $\rho^{n-1}(x)(1-|x|)^{-1/n-\epsilon} \in L^1(\mathbf{B}^n)$. One can then repeat the proof of Theorem 5.2 with $\rho^n(x)$ replaced by $\rho^{n-1}(x)(1-|x|)^{-1/n-\epsilon}$. This argument gives the desired improvement.

6. Geometry of $(\mathbf{B}^n, d_\rho)$.

In this section we define the boundary of $\mathbf{B}^n$ relative to the metric ρ and show that the geometry of our metric space resembles that of a simply connected plane domain.

It follows from Lemma 4.1 that every d_ρ-Cauchy sequence (x_k) in $\mathbf{B}^n$ is also a Cauchy sequence in the euclidean sense. For two d_ρ-Cauchy sequences (x_k) and (y_k)

$$(x_k) \sim (y_k) \iff \lim_{k\to\infty} d_\rho(x_k, y_k) = 0$$

defines an equivalence relation. It follows from HI(A) that $\mathbf{B}^n$ can be identified with a subset of the set $\mathcal{E}$ of the equivalence classes.

The metric d_ρ can be extended to $\mathcal{E}$ in the usual way: If (x_k) and (y_k) represent $x, y \in \mathcal{E}$, then $d_\rho(x,y) = \lim_{k\to\infty} d_\rho(x_k, y_k)$. The ρ-boundary $\partial_\rho \mathbf{B}^n$ of $\mathbf{B}^n$ is the set of all equivalence classes of d_ρ-Cauchy sequences (x_k) for which $(x_k) \to \partial\mathbf{B}^n$. Then $\mathcal{E} = \mathbf{B}^n \cup \partial_\rho \mathbf{B}^n$.

Proposition 6.1. *There exists a natural identification of $\partial_\rho \mathbf{B}^n$ with the set F of all $\zeta \in \partial\mathbf{B}^n$ for which* $\text{length}_\rho([0,\zeta)) < \infty$.

Proof. Every d_ρ-Cauchy sequence not converging to a point in $\mathbf{B}^n$ converges in the euclidean sense to a point $\zeta \in F$ by Theorem 4.2. Equivalent sequences converge to the same point. This defines a map from $\partial_\rho \mathbf{B}^n$ into F.

This map is surjective, since every sequence (x_k) on a radius $[0,\zeta)$, $\zeta \in F$, with $(x_k) \to \zeta$ in the euclidean sense forms a d_ρ-Cauchy sequence. Finally, the map is also injective as follows from Theorem 4.2. □

The following two results (Proposition 6.2 and Theorem 6.3) describe the geometry of our metric space. For $x \in \mathbf{B}^n$ we define $r_x = \rho(x)(1-|x|)$. By HI(A) r_x is comparable to $\text{diam}_\rho(B_x)$.

The first result shows that r_x is essentially the distance of x to the ρ-boundary.

Proposition 6.2. *There is a constant $C(A,B,n) \geq 1$ such that*

$$\frac{1}{C} \leq \frac{r_x}{\text{dist}_\rho(x, \partial_\rho \mathbf{B}^n)} \leq C \quad \textit{for all} \quad x \in \mathbf{B}^n.$$

Proof. The upper bound follows from HI(A).

Define $d = \text{dist}_\rho(x, \partial_\rho \mathbf{B}^n)$. To prove an inequality in the other direction let $E = B_x$, $F = \partial\mathbf{B}^n$, and consider the family Γ of all curves joining E and F in $\mathbf{B}^n$. Every curve $\gamma \in \Gamma$ has ρ-length at least $L = d - \delta$, where $\delta = \text{diam}_\rho(E) \leq A r_x$.

If $L < \delta$, then $d \leq 2\delta \leq 2A r_x$.

If $L \geq \delta$, then we can apply Lemma 3.2 and combine this with the modulus estimate (3.1) to get an upper bound for L/δ only depending on A, B, n. □

The following theorem is an analog of the following geometric property of a simply connected region Ω in the plane. If x is a point on a hyperbolic geodesic in Ω ending in two boundary points, then the distances of x to the two parts of the boundary of Ω lying on different sides of the geodesic are roughly the same.

Theorem 6.3. *There exists a constant $C(A,B,n)>0$ with the following property. Suppose $\zeta \in \partial \mathbf{B}^n$, $x \in (0,\zeta)$ and γ is any curve in $\mathbf{B}^n$ connecting $[0,x)$ and (x,ζ). Then $\gamma \cap B_\rho(x, C\,\mathrm{dist}_\rho(x,\partial_\rho \mathbf{B}^n)) \neq \emptyset$.*

Proof. Set $\delta = \mathrm{dist}_\rho(x,\partial_\rho \mathbf{B}^n)$ and consider the family Γ of all curves joining $E = B_\rho(x,\delta/2)$ and $F=\gamma$. We may assume that γ does not meet $B_\rho(x,\frac{3}{2}\delta)$.

Note that $\mathrm{diam}(E)$ is comparable to $(1-|x|)$. Since γ does not meet E and connects $(0,x)$ and (x,ζ), $\mathrm{diam}(\gamma)$ cannot be much smaller than $(1-|x|)$. Since $\mathrm{dist}(E,F) \leq (1-|x|)$, from (3.1) we get a positive lower bound for $\mathrm{mod}\,\Gamma$ only depending on A and n.

On the other hand, every curve in Γ has ρ-length at least $L = \mathrm{dist}_\rho(x,\gamma) - \delta/2 \geq \delta \geq \mathrm{diam}_\rho(E)$. Combining the lower modulus estimate with Lemma 3.2 now gives an upper bound for L/δ only depending on A, B and n. This proves the theorem. $\square$

7. Hausdorff dimension of the boundary.

In this section we study the size of the boundary of our metric space in terms of the Hausdorff dimension. For subsets E of $\mathbf{B}^n \cup \partial_\rho \mathbf{B}^n$ we denote by $\dim_\rho E$ the Hausdorff dimension of E with respect to d_ρ, that is

$$\dim_\rho E = \inf\{p>0: \text{ For each } \epsilon>0 \text{ there is a countable covering}$$

$$E \subseteq \bigcup_{k\in\mathbf{N}} B_\rho(x_k,r_k) \text{ with } \sum_{k=1}^{\infty} r_k^p < \epsilon\}$$

and with the usual convention $\dim_\rho E = \infty$ if there is no such p. The Hausdorff dimension of sets $E \subseteq \mathbf{R}^n$ with respect to the euclidean metric is denoted by $\dim E$.

By Proposition 6.1 and the Radial Limit theorem 4.4 we have $\dim \partial_\rho \mathbf{B}^n = n-1$ and even $\dim(\partial \mathbf{B}^n \setminus \partial_\rho \mathbf{B}^n) = 0$. However, it is more natural to study the size of $\partial_\rho \mathbf{B}^n$ in its intrinsic metric d_ρ, that is to study $\dim_\rho \partial_\rho \mathbf{B}^n$.

Proposition 7.1. *There exists a constant $\epsilon(B,n)>0$ such that*

$$\dim_\rho \partial_\rho \mathbf{B}^n \geq \epsilon.$$

On the other hand, there is a density ρ with $\dim_\rho \partial_\rho \mathbf{B}^2 = \infty$.

Proof. As mentioned above, we have $\dim \partial_\rho \mathbf{B}^n = n-1$. The identity map

$$\mathrm{id}\colon (\mathbf{B}^n \cup \partial_\rho \mathbf{B}^n, d_\rho) \to (\mathbf{B}^n \cup \partial_\rho \mathbf{B}^n, d)$$

is Hölder continuous with exponent α by Lemma 4.1, where $\alpha = \alpha(B, n)$. It follows that $\dim_\rho \partial_\rho \mathbf{B}^n \geq (n-1)\alpha$.

Our density ρ with $\dim_\rho \partial_\rho \mathbf{B}^2 = \infty$ will be the spherical derivative (cf. Section 2.2 b) of a conformal map $f\colon \mathbf{B}^2 \to \overline{\mathbf{C}}$. The image $f(\mathbf{B}^2)$ is the complement in $\overline{\mathbf{C}}$ of a continuum $H \subseteq \mathbf{R}^2$ consisting of line segments centered at the origin. This set H is constructed as follows.

Given a sequence (k_j) of positive integers, let $p_j = k_1 \cdot \ldots \cdot k_j$ for $j \in \mathbf{N}$. Let $I(j,l)$ be the line segment $[-2^{-j} e^{\pi i l/p_j}, 2^{-j} e^{\pi i l/p_j}]$ and set $H = \bigcup_{j=1}^{\infty} \bigcup_{l=1}^{p_j} I(j,l)$. Any conformal map $f\colon \mathbf{B}^2 \to \overline{\mathbf{C}} \setminus H$ has a continuous extension to $\overline{\mathbf{B}^2}$. It follows from the Gehring-Hayman theorem that the image of every radius in $\mathbf{B}^2$ has finite spherical length. Therefore, $\partial_\rho \mathbf{B}^2 = \partial \mathbf{B}^2$ and $E = f^{-1}(0)$ is a compact subset of $\partial \mathbf{B}^2$. The $\frac{3}{4} 2^{-j}$-neighborhood of E in the metric d_ρ which corresponds to the internal spherical metric of $\overline{\mathbf{C}} \setminus H$ has $2p_j$ components of mutual d_ρ-distance at least 2^{-j-1}, and each of these components contains precisely k_{j+1} components of the $\frac{3}{4} 2^{-j-1}$-neighborhood of E. It easily follows that $\dim_\rho \partial_\rho \mathbf{B}^2 = \infty$ if k_j tends to ∞ for $j \to \infty$. $\square$

Using $\dim(\partial \mathbf{B}^n \setminus \partial_\rho \mathbf{B}^n) = 0$, the above proof shows that $\dim_\rho(\partial_\rho \mathbf{B}^n \cap E) \geq \alpha \dim E$ for every set $E \subseteq \partial \mathbf{B}^n$, and in particular that $\dim_\rho(\partial_\rho \mathbf{B}^n \setminus E) \geq C$ if $E \subseteq \partial \mathbf{B}^n$ has Lebesgue $(n-1)$-measure 0. This is best possible, even for densities ρ associated with Jacobians of quasiconformal maps. For conformal maps, one has better results [**25**].

Next we show that an "essential" part of $\partial_\rho \mathbf{B}^n$ has ρ-Hausdorff dimension less or equal to n. In the conformal or quasiconformal setting the fact that the boundary has dimension at most n is obvious as the surrounding space has dimension n. In our setting it is not even clear that there is *any* bound on the ρ-dimension of "large" parts of the boundary, as $\dim_\rho \partial_\rho \mathbf{B}^n = \infty$ is possible.

Theorem 7.2. *For every $\alpha > 0$ there is a set $E \subseteq \partial_\rho \mathbf{B}^n$ with $\dim E \leq \alpha$ and*

$$\dim_\rho(\partial_\rho \mathbf{B}^n \setminus E) \leq n - \epsilon.$$

Here $\epsilon > 0$ only depends on A, B, n, α.

In particular, there is a set E with $\dim E = 0$ such that $\dim_\rho(\partial_\rho \mathbf{B}^n \setminus E) \leq n$.

The key to this result is the growth estimate of Theorem 5.2, leading to Hölder continuity on "most" radii. In fact, there is a set E of dimension at most α such that $\partial\mathbf{B}^n \setminus E$ can be decomposed into countably many sets S such that

$$\rho(t\zeta) \le C(1-t)^{-1+\alpha/n} \quad \text{for all} \quad \zeta \in S,\ 0 \le t < 1,$$

where C depends on S only. We will show $\dim_\rho S \le n - C'$ for each such set S.

Note that id: $(\mathbf{B}^n \cup \partial_\rho \mathbf{B}^n, d) \to (\mathbf{B}^n \cup \partial_\rho \mathbf{B}^n, d_\rho)$ is $\alpha/n-$Hölder continuous on S (integrate ρ over hyperbolic geodesics). But this only leads to the estimate $\dim_\rho S \le n(n-1)/\alpha$. To improve on this trivial estimate, we adopt the proof of a corresponding result for quasiconformal mappings from [**23**] to our setting. The rest of this section is devoted to proving Theorem 7.2.

In the following we denote by $\mathcal{W}$ a fixed decomposition of $\mathbf{B}^n$ into Whitney cubes, cf. [**30**]. This means that $\mathcal{W}$ is a family of closed dyadic cubes $Q \subseteq \mathbf{B}^n$ with pairwise disjoint interiors so that $\bigcup_{Q\in\mathcal{W}} Q = \mathbf{B}^n$ and

$$\operatorname{diam}(Q) \le \operatorname{dist}(Q, \partial\mathbf{B}^n) \le 4\operatorname{diam}(Q) \quad \text{for} \quad Q \in \mathcal{W}.$$

If $Q \in \mathcal{W}$, then we denote by x_Q the center of Q and put $r_Q = \rho(x_Q)(1-|x_Q|)$.

The following facts are straightforward to verify. There are constants C_1, C_2, C_3 only depending on A so that for $Q \in \mathcal{W}$ and $x \in Q$

$$\begin{aligned}(1/C_1) r_Q &\le \operatorname{diam}_\rho(Q) \le C_1 r_Q,\\ (1/C_2) r_Q^n &\le \mu_\rho(Q) \le C_2 r_Q^n,\\ (1/C_3) r_Q &\le r_x \le C_3 r_Q.\end{aligned}$$

In the next theorem we use χ_E for the characteristic function of a set E. This result will be our substitute for the usual estimates on Marcinkiewicz integrals (cf. [**5**], [**20**]).

Proposition 7.3. *If $\lambda \ge 1$, then there are constants C_1, C_2, ϵ only depending on A, B, n, λ with the following property.*

Let $\mathcal{W}' \subseteq \mathcal{W}$ be a set of Whitney cubes with $M = \sum_{Q\in\mathcal{W}'} r_Q^n < \infty$, and define $S = \bigcup_{Q\in\mathcal{W}'} B_\rho(x_Q, \lambda r_Q)$ and $T_\lambda(x) = \sum_{Q\in\mathcal{W}'} \chi_{B_\rho(x_Q,\lambda r_Q)}(x)$ for $x \in \mathbf{B}^n$.

Then $\int_S \exp(\epsilon T_\lambda)\, d\mu_\rho \le C_1 M$ *and* $\mu_\rho(\{x \in \mathbf{B}^n : T_\lambda(x) \ge t\}) \le C_2 M \exp(-\epsilon t)$ *for* $t > 0$.

Proof. We will deduce the proposition from Theorem 9.6 in the appendix. See the appendix for the formulation of the conditions A_1–A_4.

We use the metric $d = d_\rho$ and the measure $\mu = \mu_\rho$. It follows from HI(A) that there exists a number $c_1(A, n) > 0$ such that $B_\rho(x_Q, c_1 r_Q)$ is contained in the interior of Q for $Q \in \mathcal{W}$. Then the set $\mathcal{B}' = \{B_\rho(x_Q, c_1 r_Q) : Q \in \mathcal{W}\}$ consists of pairwise disjoint ρ-balls.

Since $Q \subseteq B_{x_Q}$, there exist constants $c_2(A, n)$, $c_3(A, n) > 0$ such that

$$c_2 r_Q^n \le \mu_\rho(B_\rho(x_Q, c_1 r_Q)) \le c_3 r_Q^n \quad \text{for} \quad Q \in \mathcal{W}. \tag{7.1}$$

By VG(B) there exists a constant $c_4(B, \lambda) > 0$ such that

$$\mu_\rho(B_\rho(x_Q, 5\lambda r_Q)) \le c_4 r_Q^n \quad \text{for} \quad Q \in \mathcal{W}.$$

Put $\lambda' = \lambda / c_1$. It follows that $\mathcal{B}'$ and any subset of $\mathcal{B}'$ satisfy A_2 and A_3 with $c = c_4/c_2$ and $\lambda = \lambda'$. If $\mathcal{W}'$ is any subset of $\mathcal{W}$ with $M = \sum_{Q \in \mathcal{W}'} r_Q^n < \infty$, then let $\mathcal{B} = \{B_\rho(x_Q, c_1 r_Q) : Q \in \mathcal{W}'\}$. Since $\sup_{Q \in \mathcal{W}'} r_Q \le M^{1/n}$ the set $\mathcal{B}$ satisfies A_1. By (7.1) we have

$$c_2 M \le \mu_\rho\Big(\bigcup_{B \in \mathcal{B}} B\Big) \le c_3 M.$$

In particular, $\mathcal{B}$ satisfies A_4. The statement now follows from Theorem 9.6. □

For $\lambda \ge 1$ and $x \in \mathbf{B}^n$ define

$$T_\lambda^{rad}(x) = \sum_{Q \in \mathcal{W}, Q \cap [0,x] \ne \emptyset} \chi_{B_\rho(x_Q, \lambda r_Q)}(x).$$

Roughly speaking, T_λ^{rad} counts how many Whitney cubes along the segment $[0, x]$ are not much further from x than from $\partial_\rho \mathbf{B}^n$. We first show that $T_\lambda^{rad}(x)$ is comparable to the total number of Whitney cubes that meet $[0, x]$, if $\rho(x)$ is not too large.

Lemma 7.4. *For each* $\alpha \in (0, 1)$ *and* $M > 0$ *there are constants* $\lambda(A, n, \alpha) \ge 1$, $C_1(A, n, \alpha)$ *and* $C_2(A, n, \alpha, M)$ *such that for all* $x \in \mathbf{B}^n$ *with*

$$\rho(x) \le M \rho(0)(1 - |x|)^{-\alpha}$$

we have

$$T_\lambda^{rad}(x) \geq C_1 \log\left(\frac{1}{1-|x|}\right) - C_2.$$

Proof. In addition to the hypotheses we may assume $|x| > 1/2$. Then no cube $Q \in \mathcal{W}$ contains both x and the origin. So for some number $N \in \mathbf{N}$ there exist distinct cubes $Q_1, \ldots, Q_{N+1} \in \mathcal{W}$ with the following properties: $0 \in Q_1$, $x \in Q_{N+1}$, $Q_j \cap [0,x] \neq \emptyset$ for $j \in \{1, \ldots, N+1\}$, $Q_j \cap Q_{j+1} \neq \emptyset$ and $x \notin Q_j$ for $j \in \{1, \ldots, N\}$.

The number N is comparable to the hyperbolic distance of x from the origin. This implies that there there exist constants $c_1(n)$ and $c_2(n)$ such that

$$c_1 \log\left(\frac{1}{1-|x|}\right) \leq N \leq c_2 \log\left(\frac{1}{1-|x|}\right). \tag{7.2}$$

Define $x_j = x_{Q_j}$ and $r_j = r_{x_j}$ for $j \in \{1, \ldots, N+1\}$. Since $x \notin Q_j$ and $Q_j \cap Q_{j+1} \neq \emptyset$ for $j \in \{1, \ldots, N\}$, it follows from HI(A) that there exists a constant $c_3(A, n)$ such that for all $j \in \{1, \ldots, N\}$

$$r_j \leq c_3 d_\rho(x_j, x) \quad \text{and} \quad d_\rho(x_j, x_{j+1}) \leq c_3 \min\{r_j, r_{j+1}\}.$$

This implies for $j \in \{1, \ldots, N-1\}$

$$\frac{d_\rho(x_j, x)}{d_\rho(x_{j+1}, x)} \leq \frac{d_\rho(x_{j+1}, x) + d_\rho(x_{j+1}, x_j)}{d_\rho(x_{j+1}, x)} \leq 1 + c_3^2.$$

Thus there exists a constant $c_4(A, n)$ such that for all $j \in \{1, \ldots, N-1\}$

$$\log\left(\frac{d_\rho(x_j, x)}{d_\rho(x_{j+1}, x)}\right) \leq c_4.$$

If $\lambda > c_3$ and $x \notin B_\rho(x_j, \lambda r_{x_j})$ for some $j \in \{1, \ldots, N-1\}$, then $d_\rho(x_j, x) \geq \lambda r_{x_j}$ and so

$$\frac{d_\rho(x_j, x)}{d_\rho(x_{j+1}, x)} \leq \frac{d_\rho(x_j, x)}{d_\rho(x_j, x) - c_3 r_{x_j}} \leq \frac{\lambda}{\lambda - c_3}.$$

It follows that

$$\log\left(\frac{d_\rho(x_j, x)}{d_\rho(x_{j+1}, x)}\right) \leq \log\left(\frac{\lambda}{\lambda - c_3}\right) =: \epsilon(\lambda).$$

Note that $\epsilon(\lambda) \to 0$ as $\lambda \to \infty$.

We get

$$\log\left(\frac{d_\rho(x_1,x)}{d_\rho(x_N,x)}\right) = \sum_{j=1}^{N-1}\log\left(\frac{d_\rho(x_j,x)}{d_\rho(x_{j+1},x)}\right) \le c_4 T_\lambda^{rad}(x) + \epsilon(\lambda)N.$$

Since $0 \in Q_1$ and $x \notin Q_1$ we have for some $c_5(A,n)$

$$d_\rho(x_1,x) \ge c_5 r_0 = c_5\rho(0).$$

Moreover, since $x \in Q_{N+1}$ and $Q_N \cap Q_{N+1} \neq \emptyset$ we have for some $c_6(A,n)$

$$d_\rho(x_N,x) \le c_6\rho(x)(1-|x|) \le c_6 M\rho(0)(1-|x|)^{1-\alpha}.$$

Here we used the assumption $\rho(x) \le \rho(0)M(1-|x|)^{-\alpha}$. With some constant $c_7(A,n,M)$ we obtain

$$(1-\alpha)\log\left(\frac{1}{1-|x|}\right) \le c_4 T_\lambda^{rad}(x) + \epsilon(\lambda)N + c_7.$$

If we choose $\lambda(A,n,\alpha) \ge 1$ sufficiently large, we have by (7.2)

$$\epsilon(\lambda)N \le \tfrac{1}{2}(1-\alpha)\log\left(\frac{1}{1-|x|}\right).$$

For such λ we get

$$\tfrac{1}{2}(1-\alpha)\log\left(\frac{1}{1-|x|}\right) \le c_4 T_\lambda^{rad}(x) + c_7.$$

The lemma follows. $\square$

Next we show that those points $\zeta \in \partial_\rho \mathbf{B}^n$ for which $\rho(t\zeta) = O((1-t)^{-\alpha})$ (with some positive $\alpha < 1$) can be approached by Whitney cubes such that the ρ-distance of each cube to ζ is comparable to the ρ-size of this cube. In the case of conformal mappings, this is related to a result of Gerasch [**13**], as extended in [**23**], and to geometric consequences of Hölder continuity as in [**3**], [**20**], [**29**].

Lemma 7.5. *If $\alpha \in (0,1)$ and $\lambda > \frac{1}{1-\alpha}$, then for all $\zeta \in \partial\mathbf{B}^n$ with*

$$\rho(t\zeta) = O((1-t)^{-\alpha}) \qquad \text{as } t \to 1$$

there exists a sequence $(t_k) \to 1$ with $\zeta \in B_\rho(t_k\zeta, \lambda r_{t_k\zeta})$ for $k \in \mathbf{N}$.

Proof. Suppose this is not the case. Then for some $\zeta \in \partial\mathbf{B}^n$ and $s_0 \in [0,1)$ we have $\zeta \notin B_\rho(t\zeta, \lambda r_{t\zeta})$ for all $t \in [s_0, 1)$. It follows that

$$\frac{d}{dt}\log(\mathrm{length}_\rho([t\zeta,\zeta))) = -\frac{\rho(t\zeta)}{\mathrm{length}_\rho([t\zeta,\zeta))} \geq -\frac{1}{\lambda(1-t)} \quad \text{for} \quad t \in [s_0, 1).$$

Integration from s_0 to t leads to

$$\mathrm{length}_\rho([t\zeta,\zeta)) \geq c_1(1-t)^{1/\lambda} \quad \text{for} \quad t \in [s_0, 1),$$

where $c_1 = (1-s_0)^{-1/\lambda}\,\mathrm{length}_\rho([s_0\zeta,\zeta))$.

On the other hand, integration of the upper bound for $\rho(t\zeta)$ shows that

$$\mathrm{length}_\rho([t\zeta,\zeta)) = O((1-t)^{1-\alpha}) \text{ as } t \to 1.$$

Since $1/\lambda < 1-\alpha$, this leads to a contradiction. $\square$

If a quasiconformal map on $\mathbf{B}^n$ is α-Hölder continuous, then its Jacobian has better integrability properties than expected. In this case Hölder continuity is essentially equivalent to a growth estimate for the averaged Jacobian (cf. [**2**]).

The following lemma can be regarded as a similar higher integrability result.

Lemma 7.6. *For $\alpha \in (0,1)$ there exists $\epsilon(A,B,n,\alpha) > 0$ with the following property.*

Suppose $M > 0$ and $E = \{\zeta \in \partial\mathbf{B}^n : \rho(t\zeta) \leq M\rho(0)(1-t)^{-\alpha}$ for $t \in [0,1)\}$. Let $\mathcal{W}_1$ be the set of all cubes $Q \in \mathcal{W}$ for which there exists a point $\zeta \in E$ such that $Q \cap [0,\zeta) \neq \emptyset$. Then

$$\sum_{Q \in \mathcal{W}_1} \frac{r_Q^n}{(1-|x_Q|)^\epsilon} < \infty.$$

Proof. Let $\mathcal{W}_2$ be the collection of all cubes $\tilde{Q} \in \mathcal{W}$ for which there exists $x \in \bigcup_{Q\in\mathcal{W}_1} Q$ with $\tilde{Q} \cap [0,x] \neq \emptyset$. Obviously, $\mathcal{W}_1 \subseteq \mathcal{W}_2$. For each $\zeta \in E$ we have

$$\mathrm{length}_\rho([0,\zeta)) = \int_0^1 \rho(t\zeta)\,dt \leq M\rho(0)\int_0^1 (1-t)^{-\alpha}\,dt = \frac{M\rho(0)}{1-\alpha}. \tag{7.3}$$

It is clear that for some $\lambda_0 \in (0,1)$ we have

$$F = \bigcup_{Q \in \mathcal{W}_2} Q \subseteq \bigcup_{\zeta \in E} \mathrm{Cone}(\zeta, \lambda_0, 1).$$

So from (7.3) and (4.1) we get $\mathrm{diam}_\rho(F) < \infty$. Thus $\mu_\rho(F) < \infty$ by VG(B) or equivalently $\sum_{Q \in \mathcal{W}_2} r_Q^n < \infty$.

For each $x \in Q \in \mathcal{W}_1$ there exist $\zeta \in E$ and $y \in [0, \zeta) \cap Q$. Hence

$$\rho(x) \le A\rho(y) \le AM\rho(0)(1 - |y|)^{-\alpha} \le 3^\alpha AM\rho(0)(1 - |x|)^{-\alpha}.$$

By Lemma 7.5 there are constants $\lambda(A, n, \alpha) \ge 1$, $C_1(A, n, \alpha)$, $C_2(A, n, \alpha, M)$ such that

$$T_\lambda^{rad}(x) \ge C_1 \log\left(\frac{1}{1 - |x_Q|}\right) - C_2 \quad \text{for} \quad x \in Q \in \mathcal{W}_1.$$

Define T_λ as in Proposition 7.3 for $\mathcal{W}' = \mathcal{W}_2$. Then $T_\lambda(x) \ge T_\lambda^{rad}(x)$ for $x \in \bigcup_{Q \in \mathcal{W}_1} Q$ by definition of $\mathcal{W}_2$. Furthermore, for $S' = \bigcup_{Q \in \mathcal{W}_2} B_\rho(x_Q, \lambda r_Q)$ and some $\epsilon_1(A, B, n) > 0$ we have

$$\int_{S'} \exp(\epsilon_1 T_\lambda)\, d\mu_\rho < \infty.$$

If we choose $c_1(A, n) \le 1$ small enough, then $B_\rho(x_Q, c_1 r_Q)$ is contained in the interior of Q for $Q \in \mathcal{W}_2$ and so the ρ-balls $B_\rho(x_Q, c_1 r_Q)$, $Q \in \mathcal{W}_2$, are pairwise disjoint. Moreover, there exists a constant $c_2(A, n)$ such that $\mu_\rho(B_\rho(x_Q, c_1 r_Q)) \ge c_2 r_Q^n$ for $Q \in \mathcal{W}_2$. Therefore, with constants $c_3(A, B, n, \alpha, M)$ and $\epsilon(A, B, n, \alpha) = \epsilon_1 C_1$

$$\begin{aligned}
\infty > \int_{S'} \exp(\epsilon_1 T_\lambda)\, d\mu_\rho &\ge \sum_{Q \in \mathcal{W}_1} \int_{B_\rho(x_Q, c_1 r_Q)} \exp(\epsilon_1 T_\lambda)\, d\mu_\rho \\
&\ge \exp(-\epsilon_1 C_2) \sum_{Q \in \mathcal{W}_1} \frac{\mu_\rho(B_\rho(x_Q, c_1 r_Q))}{(1 - |x_Q|)^{\epsilon_1 C_1}} \\
&\ge c_3 \sum_{Q \in \mathcal{W}_1} \frac{r_Q^n}{(1 - |x_Q|)^\epsilon}.
\end{aligned}$$

The claim follows. □

Proof of Theorem 7.2. By Theorem 5.2 there exists a set $E \subseteq \partial \mathbf{B}^n$ with $H^\alpha(E) = 0$ such that for all $\zeta \in \partial \mathbf{B}^n \setminus E$

$$\rho(t\zeta) = o((1 - t)^{-1 + \alpha/n}) \quad \text{as} \quad t \to 1.$$

It follows that $\partial\mathbf{B}^n \setminus E = \bigcup_{k\in\mathbf{N}} E_k$, where

$$E_k = \{\zeta \in \partial\mathbf{B}^n : \rho(t\zeta) \le k\rho(0)(1-t)^{-1+\alpha/n} \text{ for } t \in [0,1)\}, \quad k \in \mathbf{N}.$$

We will show $\dim_\rho(E_k) \le n-\epsilon$ for some $\epsilon(A,B,n,\alpha)$ independent of $k \in \mathbf{N}$. This implies $\dim_\rho(\partial\mathbf{B}^n \setminus E) \le n-\epsilon$.

For fixed $k \in \mathbf{N}$ define $\mathcal{W}_1$ as the collection of all cubes $Q \in \mathcal{W}$ for which there exist $\zeta \in E_k$ with $Q \cap [0,\zeta) \neq \emptyset$. By Lemma 7.6 there exists $\epsilon_1(A,B,n,\alpha)$ such that

$$\sum_{Q\in\mathcal{W}_1} \frac{r_Q^n}{(1-|x_Q|)^{\epsilon_1}} < \infty.$$

By Theorem 5.1 there exist constants $c_1(\rho)$ and $c_2(B,n)$ such that

$$(1-|x_Q|)^{c_2} \le c_1 r_Q \quad \text{for} \quad Q \in \mathcal{W}.$$

With the constant $\epsilon = \epsilon_1/c_2$ which only depends on A,B,n,α and not on k, we then have $\sum_{Q\in\mathcal{W}_1} r_Q^{n-\epsilon} < \infty$. Thus, to prove $\dim_\rho(E_k) \le n-\epsilon$ it it enough to show that there exists $\lambda(A,n,\alpha) \ge 1$ such that

$$E_k \subseteq \bigcup_{Q\in\mathcal{W}_1^\delta} B_\rho(x_Q, \lambda r_Q) \tag{7.4}$$

for every $\delta > 0$. Here $\mathcal{W}_1^\delta$ is the set of all $Q \in \mathcal{W}_1$ for which $r_Q \le \delta$.

By Lemma 7.5 there exists $\lambda'(n,\alpha) \ge 1$ so that for each $\zeta \in E_k$ there exists a sequence (x_j) on the radius $[0,\zeta)$ with $\lim_{j\to\infty} x_j = \zeta$ and $\zeta \in B_\rho(x_j, \lambda' r_{x_j})$ for $j \in \mathbf{N}$. For each $j \in \mathbf{N}$ there exists a cube $Q_j \in \mathcal{W}_1$ with $x_j \in Q_j$. For some constant $c_3(A,n)$ we have

$$d_\rho(x_j, x_{Q_j}) \le c_3 r_{Q_j} \quad \text{and} \quad (1/c_3) r_{Q_j} \le r_{x_j} \le c_3 r_{Q_j}, \quad j \in \mathbf{N}.$$

It follows that for the constant $\lambda = 2c_3\lambda'$ which only depends on A,n,α

$$\zeta \in B_\rho(x_j, \lambda' r_{x_j}) \subseteq B_\rho(x_{Q_j}, \lambda r_{Q_j}) \quad \text{for} \quad j \in \mathbf{N}.$$

The upper bound on $\rho(t\zeta)$ for $\zeta \in E_k$ implies $\lim_{j\to\infty} r_{x_j} = 0$. Thus $\lim_{j\to\infty} r_{Q_j} = 0$ and we conclude that for every $\delta > 0$ the inclusion (7.4) holds. This finishes the proof of the first part of the theorem.

To prove the second part, choose for each $j \in \mathbf{N}$ a set $F_j \subseteq \partial\mathbf{B}^n$ with $\dim(F_j) \le 1/j$ and $\dim_\rho(\partial_\rho\mathbf{B}^n \setminus F_j) \le n$. This is possible by the first part of the proof. If we define $E = \bigcap_{l\in\mathbf{N}} \left(\bigcup_{j\ge l} F_j \right)$, then $\dim(E) = 0$ and $\dim_\rho(\partial_\rho\mathbf{B}^n \setminus E) \le n$. $\square$

8. Integral means.

For $t \in [0,1)$ and $0 < p \leq n$ let

$$I_p(\rho, t) = \frac{1}{\omega_{n-1}} \int_{\partial \mathbf{B}^n} \rho(t\zeta)^p \, d\sigma_{n-1}(\zeta).$$

Here integration is with respect to $(n-1)$-dimensional surface measure σ_{n-1} on $\partial \mathbf{B}^n$. If $\mu_\rho(\mathbf{B}^n) < \infty$ it is possible to obtain an estimate on the asymptotic behavior of the integral mean $I_p(\rho, t)$ for $t \to 1$.

For $t \in [0,1)$ consider a shell, e.g., $S_t = \{x \in \mathbf{B}^n : t - \frac{1}{5}(1-t) \leq |x| \leq t + \frac{1}{5}(1-t)\}$ around the sphere $\{x \in \mathbf{B}^n : |x| = t\}$. It follows from HI$(A)$ that for some $c_1(A, n)$

$$I_n(\rho, t) \leq \frac{c_1}{1-t} \int_{S_t} \rho^n \, dm_n \leq \frac{c_1}{1-t} \mu_\rho(\mathbf{B}^n),$$

and so $I_n(\rho, t) = O\big(\frac{1}{1-t}\big)$ for $t \to 1$. Hölder's inequality implies

$$I_p(\rho, t) \leq I_n(\rho, t)^{p/n} = O((1-t)^{-p/n}) \quad \text{for} \quad t \to 1. \tag{8.1}$$

The assumption $\mu_\rho(\mathbf{B}^n) < \infty$ is true in particular if $n = 2$ and $\rho = |f'|$ for a bounded conformal map $f \colon \mathbf{B}^2 \to \mathbf{C}$. In this case it was shown in [**6**] that for $0 < p < n$ the exponent $-p/n$ in (8.1) can be replaced by a larger exponent independent of f. This improvement on the obvious estimate (8.1) reveals the more hidden specifics of the growth behavior of the derivative of a bounded conformal map.

This feature remains true for our densities ρ as the following Theorem 8.1 shows.

Theorem 8.1. *Suppose $\mu_\rho(\mathbf{B}^n) < \infty$. Then for every $0 < p < n$ there exists a number $\epsilon(A, B, n, p) > 0$ such that*

$$I_p(\rho, t) = O((1-t)^{-p/n+\epsilon}) \quad \textit{for} \quad t \to 1.$$

The proof uses ideas developed in [**5**]. The crucial point is that the function T_λ introduced in Section 7 is a suitable replacement for the Marcinkiewicz integrals employed in [**5**].

Lemma 8.2. *For $t \in [0,1)$ denote by $\mathcal{W}^t$ the set of all cubes $Q \in \mathcal{W}$ for which $Q \cap \{x \in \mathbf{B}^n : |x| = t\} \neq \emptyset$.*

a) $\frac{1}{2}(1-t) \leq 1-|x| \leq 2(1-t)$ *for* $x \in Q \in \mathcal{W}^t$, $t \in [0,1)$.

b) Let $\#\mathcal{W}^t$ *be the cardinality of* $\mathcal{W}^t$. *There exists a constant* $C_1(n)$ *so that*

$$\#\mathcal{W}^t \leq C_1(1-t)^{1-n} \quad \text{for} \quad t \in [0,1).$$

c) For $0 < p \leq n$ *there exists a constant* $C_2(A,n,p)$ *with*

$$I_p(\rho,t) \leq C_2(1-t)^{n-p-1} \sum_{Q \in \mathcal{W}^t} r_Q^p \quad \text{for} \quad t \in [0,1).$$

Proof. The proofs of a) and b) are straightforward.

For the proof of c) first estimate $I_p(\rho,t)$ in terms of an integral of ρ^p over a thin shell S_t (cf. the discussion above) around the sphere $\{x \in \mathbf{B}^n : |x| = t\}$. If we blow up the cubes in $\mathcal{W}^t$ by a fixed factor, the union of the the blown up cubes will contain the shell. The desired estimate then follows from HI(A). The details are left to the reader. $\square$

Proof of Theorem 8.1. The assumption $\mu_\rho(\mathbf{B}^n) = \sum_{Q \in \mathcal{W}} \mu_\rho(Q) < \infty$ is equivalent to $\sum_{Q \in \mathcal{W}} r_Q^n < \infty$.

Let $t \in [0,1)$ and $\epsilon_1 = \frac{1}{2}(1-1/n)$. We split the set $\mathcal{W}^t$ introduced in Lemma 8.2 into the set $\mathcal{W}_1^t = \{Q \in \mathcal{W}^t : r_Q > \rho(0)(1-t)^{1-1/n-\epsilon_1}\}$ of "good" cubes and the set $\mathcal{W}_2^t = \{Q \in \mathcal{W}^t : r_Q \leq \rho(0)(1-t)^{1-1/n-\epsilon_1}\}$ of bad "cubes".

Note that

$$\#\mathcal{W}_1^t(1-t)^{n-1-n\epsilon_1} \leq \frac{1}{\rho(0)^n} \sum_{Q \in \mathcal{W}} r_Q^n$$

and so $\#\mathcal{W}_1^t \leq c_1(1-t)^{1-n+n\epsilon_1}$ for some $c_1(\rho)$. By Hölder's inequality this implies that for some $c_2(\rho,p)$

$$\sum_{Q \in \mathcal{W}_1^t} r_Q^p \leq \left(\sum_{Q \in \mathcal{W}_1^t} r_Q^n \right)^{p/n} (\#\mathcal{W}_1^t)^{1-p/n} \leq c_2(1-t)^{-(1-1/n)(n-p)+\epsilon_1(n-p)}. \tag{8.2}$$

By Lemma 8.2 a) and HI(A) there exists a constant $c_3(A,n)$ so that for $x \in Q \in \mathcal{W}_2^t$

$$\rho(x) \leq A\rho(x_Q) \leq \frac{2Ar_Q}{1-t} \leq \frac{2A\rho(0)}{(1-t)^{1/n+\epsilon_1}} \leq \frac{c_3\rho(0)}{(1-|x|)^{1/n+\epsilon_1}}.$$

Lemma 7.4 for $M = c_3$ and $\alpha = 1/n + \epsilon_1 \in (0,1)$ shows that there are constants $\lambda \geq 1$, c_4, c_5 only depending on A and n so that for each $x \in Q \in \mathcal{W}_2^t$

$$T_\lambda^{rad}(x) \geq c_4 \log\left(\frac{1}{1-t}\right) - c_5.$$

If

$$T_\lambda(x) = \sum_{Q\in\mathcal{W}} \chi_{B_\rho(x_Q,\lambda r_Q)}(x) \quad \text{for} \quad x \in \mathbf{B}^n,$$

then it follows from $T_\lambda \geq T_\lambda^{rad}$ that

$$T_\lambda(x) \geq c_4 \log\left(\frac{1}{1-t}\right) - c_5 \quad \text{for} \quad x \in \bigcup_{Q\in\mathcal{W}_2^t} Q.$$

From Proposition 7.3 we conclude that there exist constants $c_6(\rho)$, $\epsilon_2(A,B,n) > 0$ so that

$$\mu_\rho\left(\bigcup_{Q\in\mathcal{W}_2^t} Q\right) \leq c_6(1-t)^{\epsilon_2}.$$

This implies that for some $c_7(\rho) > 0$

$$\sum_{Q\in\mathcal{W}_2^t} r_Q^n \leq c_7(1-t)^{\epsilon_2}.$$

Hölder's inequality and Lemma 8.2 b) then show that for some $c_8(\rho) > 0$

$$\sum_{Q\in\mathcal{W}_2^t} r_Q^p \leq \left(\sum_{Q\in\mathcal{W}_2^t} r_Q^n\right)^{p/n} (\#\mathcal{W}_2^t)^{1-p/n} \leq c_8(1-t)^{-(1-1/n)(n-p)+\epsilon_2 p/n}. \tag{8.3}$$

From (8.2) and (8.3) we see that for $\epsilon(A,B,n,p) = \min\{\epsilon_1(n-p), \epsilon_2 p/n\} > 0$ and some $c_9(\rho)$

$$\sum_{Q\in\mathcal{W}^t} r_Q^p \leq c_9(1-t)^{-(1-1/n)(n-p)+\epsilon}.$$

Together with Lemma 8.2 c) this implies the assertion. $\square$

9. Appendix: Exponential integrability in metric measure spaces. Suppose (X, d) is a metric space and μ a σ-finite Borel measure on X. In this section all metric and measure theoretic notions will refer to the metric d and the measure μ, respectively.

We fix a number $\lambda \geq 1$ and a countable set $\mathcal{B}$ of open balls in X. The purpose of this section is to show that if the balls in $\mathcal{B}$ are pairwise disjoint, then the set where many of the balls λB, $B \in \mathcal{B}$, overlap has small measure. Here we put $\lambda B = B(x, \lambda r)$ for a ball $B = B(x, r)$. To show this we introduce the function

$$T_\lambda(x) = \sum_{B \in \mathcal{B}} \chi_{\lambda B}(x) \quad \text{for} \quad x \in X$$

and establish that T_λ is exponentially integrable under suitable conditions on $\mathcal{B}$ (cf. Theorem 9.6).

The following conditions for $\mathcal{B}$ will be relevant.

A_1: $\sup_{B \in \mathcal{B}} \operatorname{diam}(B) < \infty$.

A_2: There exists a constant $c \geq 1$ so that $0 < \mu(5\lambda B) \leq c\mu(B)$ for all $B \in \mathcal{B}$.

A_3: The balls in $\mathcal{B}$ are pairwise disjoint.

A_4: For $S = \bigcup\{B : B \in \mathcal{B}\}$ we have $\mu(S) < \infty$.

The proof of exponential integrability of T_λ runs along familiar lines. The reason for presenting the details is that our measure is not doubling and we only work with the weaker condition A_2.

Lemma 9.1. *If $\mathcal{B}$ satisfies A_1, then there exists a set $\mathcal{B}' \subseteq \mathcal{B}$ of pairwise disjoint balls with $\bigcup\{B : B \in \mathcal{B}\} \subseteq \bigcup\{5B : B \in \mathcal{B}'\}$.*

The proof is standard (cf. [**32**], p. 7).

If $f \colon X \to [0, \infty]$ is any Borel function, we denote by $M_f \colon X \to [0, \infty]$ the "maximal function" defined by

$$M_f(x) = \sum_{B \in \mathcal{B}} \left(\frac{1}{\mu(\lambda B)} \int_{\lambda B} f \, d\mu \right) \chi_B(x) \quad \text{for} \quad x \in X.$$

Lemma 9.2. *Suppose $\mathcal{B}$ satisfies A_1–A_3. Then*

$$\mu(\{x : M_f(x) > t\}) \le \frac{c}{t} \int f \, d\mu$$

for all Borel functions $f\colon X \to [0,\infty]$ and all $t > 0$.

Proof. Let $\mathcal{B}_t = \{B \in \mathcal{B} : \int_{\lambda B} f \, d\mu > t\mu(\lambda B)\}$. Since the balls in $\mathcal{B}$ are pairwise disjoint, $\{x : M_f(x) > t\} = \bigcup\{B : B \in \mathcal{B}_t\}$. Since $\mathcal{B}$ satisfies A_1, the set $\mathcal{A}_t = \{\lambda B : B \in \mathcal{B}_t\}$ also satisfies A_1. It follows from Lemma 9.1 applied to $\mathcal{A}_t$ that there exists a subset $\mathcal{B}'_t \subseteq \mathcal{B}_t$ so that $\mathcal{A}'_t = \{\lambda B : B \in \mathcal{B}'_t\}$ consists of pairwise disjoint balls and

$$\bigcup\{\lambda B : B \in \mathcal{B}_t\} \subseteq \bigcup\{5\lambda B : B \in \mathcal{B}'_t\}.$$

Using this and A_2 we get

$$\begin{aligned}\mu(\{x : M_f(x) > t\}) &\le \mu\Big(\bigcup\{5\lambda B : B \in \mathcal{B}'_t\}\Big) \le c \sum_{B \in \mathcal{B}'_t} \mu(\lambda B) \\ &\le \frac{c}{t} \sum_{B \in \mathcal{B}'_t} \int_{\lambda B} f \, d\mu = \frac{c}{t} \int_{\bigcup\{\lambda B : B \in \mathcal{B}'_t\}} f \, d\mu \le \frac{c}{t} \int f \, d\mu.\end{aligned}$$

$\square$

Lemma 9.3. *Suppose $\mathcal{B}$ satisfies A_1–A_3. Then*

$$\mu(\{x : M_f(x) > t\}) \le \frac{2c}{t} \int_{\{x : f(x) > t/2\}} f \, d\mu$$

for all Borel functions $f\colon X \to [0,\infty]$ and all $t > 0$.

Proof. By a standard procedure, this is reduced to Lemma 9.2.

For fixed $t > 0$ let $g = f\chi_{\{x : f(x) > t/2\}}$ and $h = f\chi_{\{x : f(x) \le t/2\}}$. Then $h \le t/2$ and so $\{x : M_f(x) > t\} \subseteq \{x : M_g(x) > t/2\}$. The assertion follows from Lemma 9.2 applied to g. $\square$

Lemma 9.4. *Suppose $\mathcal{B}$ satisfies A_1–A_3. Then, for $1 < p < \infty$ and any Borel function $f\colon X \to [0,\infty]$,*

$$\int (M_f)^p \, d\mu \le c \frac{2^p p}{p-1} \int f^p \, d\mu.$$

Proof. It is well-known how this statement follows from the estimate in Lemma 9.3 (cf. [**8**], p. 146). $\square$

Lemma 9.5. *Suppose $\mathcal{B}$ satisfies A_1–A_4. Then for $1 \le q < \infty$*

$$\int T_\lambda^q \, d\mu \le (2c^2 q)^q \mu(S).$$

Proof. For $q = 1$ this follows from A_2 and A_3.

For $q > 1$ let $p > 1$ be the number with $1/q + 1/p = 1$. For an arbitrary function $f\colon X \to [0, \infty]$ in $L^p(\mu)$, we have by Lemma 9.4

$$\begin{aligned}\int f T_\lambda \, d\mu &= \sum_{B \in \mathcal{B}} \int_{\lambda B} f \, d\mu \le c \sum_{B \in \mathcal{B}} \Big(\frac{1}{\mu(\lambda B)} \int_{\lambda B} f \, d\mu \Big) \mu(B) \\ &= c \int M_f \, d\mu = c \int \chi_S M_f \, d\mu \le c \Big(\int \chi_S^q \, d\mu \Big)^{1/q} \Big(\int M_f^p \, d\mu \Big)^{1/p} \\ &\le c \mu(S)^{1/q} \|M_f\|_p \le 2c^2 q \mu(S)^{1/q} \|f\|_p .\end{aligned}$$

The assertion follows from the duality of $L^p(\mu)$ and $L^q(\mu)$. □

Theorem 9.6. *Suppose $\mathcal{B}$ satisfies A_1–A_4. Let $S' = \bigcup \{\lambda B : B \in \mathcal{B}\}$ and $\epsilon = \frac{1}{12c^2}$. Then*

a) $\displaystyle\int_{S'} \exp(\epsilon T_\lambda) \, d\mu \le (1 + c)\mu(S)$,

b) $\mu(\{x : T_\lambda(x) \ge t\}) \le (1 + c)\mu(S) \exp(-\epsilon t)$ *for* $t > 0$.

Proof. To prove a) expand the exponential into a series, use $\mu(S') \le c\mu(S)$ to estimate the first term and Lemma 9.5 and the inequality $q^q \le 3^q q!$, $q \in \mathbf{N}$, for the other terms.

Since $\{x : T_\lambda(x) > 0\} = S'$ statement b) follows from a). □

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Department of Mathematics, TU Braunschweig, Pockelsstrasse 14, 38106 Braunschweig, Germany,

Department of Mathematics, University of Jyväskylä, P.O. Box 35, SF-40351 Jyväskylä, Finland,

Department of Mathematics, TU Berlin, Strasse des 17. Juni 136, 10623 Berlin, Germany