University of Washington
Conformal Invariance and Probability
S. Rohde Autumn 2011

Exercise Set 3

Problem 1. Show that for all $\kappa \neq 4$ there is $\alpha(\kappa) > 0$ such that the conformal maps $g_t^{-1} : \mathbb{H} \to \mathbb{H} \setminus K_t$ are $\alpha(\kappa)-$Hölder continuous a.s. More precisely, show that for all bounded sets $A \subset \mathbb{H}$ and all $t > 0$ there is $C = C(A, t, \omega)$ such that

$$|g_t^{-1}(z) - g_t^{-1}(w)| \leq C|z - w|^{\alpha(\kappa)}$$

for all $z, w \in A$.

Problem 2. a) Find a bijection between the set of spanning trees on a planar graph G, and the spanning trees on its dual G^*.

b) Let $G = \mathbb{D} \cap \epsilon\mathbb{Z}^2$ be a grid approximation to the disc, and let a and b be vertices of G closest to -1 and 1. Let γ be a simple path in G from a to b, chosen uniformly at random among all such simple paths. Show that γ will be "dense" when ϵ is small: For each $r > 0$,

$$P[\text{ there is } z \in \mathbb{D} \text{ such that } \gamma \cap D(z, r) = \emptyset] \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Problem 3. For each $\kappa > 0$ and each $z \in \mathbb{H}$, show that $y_t = \text{Im}(g_t(z)) \to 0$ a.s. as $t \to T_z$.

Problem 4. Show that, for $c \notin \{0, -1, -2, \cdots\}$, the hypergeometric series

$${}_2F_1(a, b, c; z) = 1 + \frac{ab}{c1!}z + \frac{a(a+1)b(b+1)}{c(c+1)2!}z^2 + \cdots$$

converges in the unit disc, satisfies the hypergeometric differential equation

$$z(1 - z)F''(z) + (c - (a + b + 1)z)F'(z) - abF(z) = 0,$$

and can be analytically continued along any curve $\gamma \subset \mathbb{C} \setminus \{0, 1\}$.

Problem 5. Let $T = T(a, b, c)$ be an (isosceles) triangle with angles $\alpha = \beta = (1 - \frac{4}{\kappa})\pi$ and $\gamma = (\frac{8}{\kappa} - 1)\pi$, and consider SLE_κ in T from a to b. Show that, for $4 < \kappa < 8$, the first intersection with the segment $[b, c]$ is uniformly distributed.

Problem 6. Show that the probability that SLE_4 passes a given point $z \in \mathbb{H}$ to the right equals $\frac{\arg z}{\pi}$.

Due date : Wednesday, December 7.