University of Washington
Conformal Invariance and Probability - Math 583
S. Rohde Spring 2011

Exercise Set 4

Problem 1. Let $\gamma : [0, T) \to \mathbb{D} \cup \{1\}$ be a simple curve with $\gamma(0) = 1$, and denote g_t the conformal map from $\mathbb{D} \setminus \gamma[0, t]$ onto $\mathbb{D}$ that is normalized by $g_t(0) = 0$ and $g_t'(0) > 0$.

a) Show that $a(t) = \log g_t'(0)$ is continuous, strictly increasing, and $a(0) = 0$. Thus there is a re-parametrization of γ such that $a(t) = t$. Assume this normalization for the rest of the problem.

b) Show that $\lambda(t) = g_t(\gamma(t))$ exists (as a limit), and is continuous.

c) Show that, for $z \in \mathbb{D} \setminus \gamma[0, t]$, the map $t \mapsto g_t(z)$ is differentiable in t and satisfies the *radial Loewner equation*

$$\frac{\partial}{\partial t} g_t(z) = -g_t(z) \frac{g_t(z) + \lambda(t)}{g_t(z) - \lambda(t)}.$$

Problem 2. If γ is a circle in $\mathbb{H}$ with center at $\mathbb{R}$, compute its driving term.

Problem 3. If $\gamma : [0, T) \to \overline{\mathbb{H}}$ is "conformally self-similar" (that is, if $g_t(\gamma)$ is similar to γ for each $t < T$), show that $\lambda(t) = a + b\sqrt{T - t}$ for constants a and b.

Problem 4. Convince yourself that the Sierpinski triangle and the Hilbert space-filling curve indicated in the handout are indeed continuous curves, and that the conformal maps onto their complements satisfy the Loewner differential equation.

Due date : Wednesday, May 18.