University of Washington
Conformal Invariance and Probability - Math 583
S. Rohde Spring 2011

Exercise Set 2

Problem 1. Given $z_0, z_1, ..., z_n$ distinct points in $\hat{\mathbb{C}}$, let $z_{n+1} = z_0$, let $\gamma_1 = [z_0, z_1]$, and inductively define the curve γ_{k+1} as the hyperbolic geodesic in $\hat{\mathbb{C}} \setminus (\gamma_1 \cup \cdots \cup \gamma_k)$ from z_k to z_{k+1} $(1 \leq k \leq n)$.

a) Show that the set of points $z_0, ..., z_n \subset \mathbb{C}^{n+1}$ for which this "makes sense" (that is, for which $z_j \notin \gamma_1 \cup \cdots \cup \gamma_k$ for all $1 \leq k < j$) is dense and of full Lebesgue measure.

b) Explain how the geodesic zipper algorithm discussed in class yields conformal maps from the unit disc onto the two domains bounded by the simple closed curve $\gamma_1 \cup \cdots \cup \gamma_n$.

Problem 2. a) Let $f : \mathbb{D} \to G$ be a conformal map with $f(0) = 0$, and define $g(z) = z\sqrt{f(z^2)/z^2}$. Prove that g is a well-defined analytic function and that f is again a conformal map.

b) For each $n = 1, 2, 3, ...$, consider the domain $G_n = \mathbb{C} \setminus \cup_{k=1}^{n} R_k$, where $R_k = \{re^{2\pi ik/n} : r \geq 1\}$ is the ray from an n-th root of unity to ∞. Determine the conformal map f_n from $\mathbb{D}$ to G_n with $f(0) = 0$, $f'(0) > 0$. Do the f_n converge locally uniformly, and if yes, what is the limit?

Problem 3. Let $0 < a < 1$. Show that the function $f(z) = (z - a)^a (z - a + 1)^{a-1}$ provides a conformal map from the slit upper half plane $\mathbb{H} \setminus [0, r_a e^{i\pi a}]$ for some $r_a > 0$.

Problem 4. Let $G \subsetneq \mathbb{C}$ be simply connected and $z \in G$. Define the *conformal radius cr* and the *hyperbolic density* λ as

$$cr_G(z) = \frac{1}{\lambda_G(z)} \equiv |\phi'(0)|,$$

where ϕ is a conformal map from $\mathbb{D}$ onto G with $\phi(0) = z$.

a) Show that

$$\lambda_{\mathbb{D}}(z) = \frac{1}{1 - |z|^2}.$$

b) Show that the differential $\lambda(z)|dz|$ is conformally invariant: If $f : G \to H$ is a conformal map, then

$$\lambda_G(z) = \lambda_H(f(z))|f'(z)|.$$

Conclude that the infimum of

$$\int_\gamma \lambda_G(z)|dz|$$

over all curves γ in G that join two given points z and w equals the hyperbolic distance of z and w in G.

c) The *quasihyperbolic distance* d_{qh} of z and w in G is defined as the infimum of

$$\int_\gamma \frac{1}{\operatorname{dist}(z, \partial G)} |dz|.$$

Show that $d_h \leq d_{qh} \leq 4 d_h$.

d) Show that

$$\lambda_G(z) = \operatorname{cap}\left(\frac{1}{\partial G - z}\right).$$

Use this to prove Jorgensens Lemma.

Due date : Monday, April 25.