Math 583E-Introduction to SLE

Scribe: John Lombard

Week 1

1 What is SLE?

1.1 Random Conformal Maps

One of the quickest ways to build an intuition for SLE is through an explicit construction using random conformal maps.

Consider a slotted half-plane denoted $\tilde{\mathbb{H}}$, with the slit prescribed by a phaser base-pointed at the origin, $e^{ai\pi}|_{0<a<1}$.

A map from the upper half plane $\mathbb{H}$ into $\tilde{\mathbb{H}}$ can be found through the usual Schwartz-Christoffel mapping, with

$$\begin{aligned} f_1 : \mathbb{H} &\rightarrow \tilde{\mathbb{H}} \\ f_1(z) &\mapsto c(z-a)^a(z-(a-1))^{1-a} \end{aligned}$$

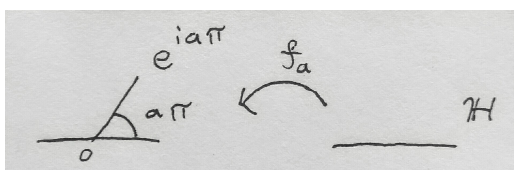

Figure 1: Example of the f map

We can implement the map into the slotted half-plane reflected over the imaginary axis, $\tilde{\mathbb{H}}^R$, similarly. We denote it $f_2(z)$.

If we now allow a uniformly random composition of these maps such that

$$f = \phi_1 \circ \cdots \circ \phi_n | \, \phi_j = f_1 \text{ or } f_2 \text{ with probability } \frac{1}{2}$$

we generate a particular example of an SLE curve.

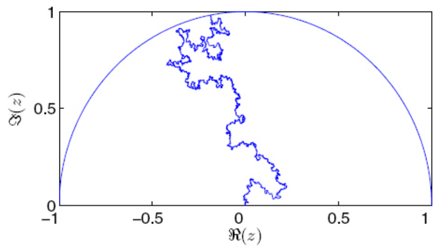

Figure 2: $SLE_{8/3}$ Example Simulation (A. Zoia, Y. Kantor, and M. Kardar)

Normalizing at infinity rather than fixing the origin, we note that under repeated iteration of these maps, the base-point undergoes a discrete random walk along the real axis. In the appropriate scaling limit which can be heuristically thought of as the naive 'look-from-afar' picture, the random motion of the base-point is described by Brownian motion.

To provide an analogy from a physicist's perspective, classes of SLE curves in the plane are to projected Brownian motion on the line as classes of circular orbits in the plane are projected to simple harmonic motion on the line. Perhaps this is a deeply flawed analogy, but I like its simplistic reduction.

1.2 Scaling Limits of Statistical Physics Lattice Models

SLE is found across a wide variety of physical systems.

The self-avoiding walk (SAW) was pioneered by Flory, a chemist studying polymer growth. The construction is built on a $\mathbb{Z}^2$ lattice, with the goal of obtaining a simple lattice path of a fixed length n. As an aside, finding the total number of such walks for fixed n is an almost intractable problem with exponential scaling; however, triangular lattices offer an exact solution. Nevetheless, these walks are conjectured to be of an SLE nature.

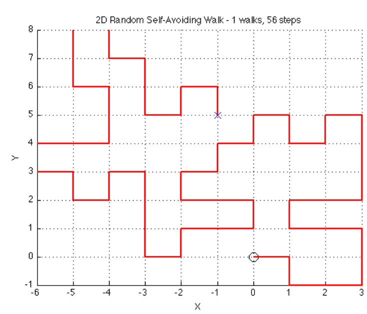

Figure 3: SAW walk example (Hayes)

The loop erased random walk (LERW) is another model quite similar to the above, wherein a graphical tree is grown greedily out to some distance by simply discarding any loops which form. This curve also has an SLE limit.

1.3 'Canonical' Measure on the Space of Planar Curves/Loops

Consider some continuous random curve from points a to b on a simply connected domain D. We can define a family of probability measures $P_{D,a,b}$ on the space of such curves γ in $\bar{D}$. Notationally, if $(a,b) \in \partial D$, we call the curve 'chordal'; otherwise for curves with $a \in \partial D$ and $b \in D$, they are known as 'radial'.

Although we need not, as curves which touch but 'bounce' off are still allowed in the construction, we can restrict to a class of simple curves for convenience. These will turn out to correspond to $SLE_{k\leq 4}$.

We note that all of the curves are conformally invariant (equivalent under locally angle preserving maps). That is, given a new setup denoted by the tuple (a', b', D'),

$$\begin{aligned} \text{if } f : D &\rightarrow D' \text{ is conformal} \\ a &\mapsto a' \\ b &\mapsto b' \end{aligned}$$

then

$$f \circ P_{D,a,b} = P_{D',a',b'}$$

In short, the measure is transported in the obvious way.

We see that we have the freedom to work in our favorite domain of choice which respects the boundary symmetries for convenience. For the radial case, we conformally map to open disks; for chordal, use the half-plane with the curves extending from the origin to infinity.

A remarkable result is the restriction property associated with $SLE_{8/3}$, wherein if we consider excising a portion of D such that our new domain is some $D' \subset D$ and simpily throw out any curves which run through the former excised portion,

$$P_{D,a,b}|_{\gamma \subset D'} = P_{D',a,b} = SLE_{8/3}$$

Fixing certain normalizations, this is the unique family of measures of D between a and b with this property.

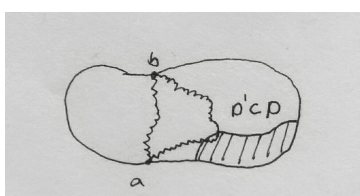

Figure 4: Illustration of restriction/excision property

2 Loewner Equation

Consider a simple curve $\gamma = [0, \infty) \to \mathbb{H} \cup \{0\}$, $\gamma(0) = 0$. Then is is easy to see that the complement space is simply connected. We choose the following notion for the definition of the curve-complement with a finite curve parameterization t:

$$H_t \equiv \mathbb{H} - \gamma[0, t]$$

By the Riemann mapping theorem, H_t , as a non-empty simply connected open strict subset of $\mathbb{C}$, is a conformal disk. We define a conformal, analytic and injective map

$$f_t := \mathbb{H} \to H_t$$

Lemma $\exists!$ conformal f_t with $f_t(\infty) = \infty$ such that

$$f_t(z) = 1 \cdot z + 0 + \frac{c}{z} + \vartheta(\frac{1}{z^2}) \text{ with } ||z|| \gg 1$$

Toward a proof, we work with disk mappings using the standard Cayley transform, a conformal map from the upper half plane of $\mathbb{C}$ onto the unit disc given by

$$W : z \to \frac{z - i}{z + i}$$

We build following mapping diagram:

$$\begin{array}{ccc} \tilde{\mathbb{H}} & \underset{W}{\to} & \tilde{D} \\ & & \uparrow \phi \\ \mathbb{H} & \underset{W}{\to} & D \end{array}$$

Small neighborhoods of 1 on the disk become very large regions in $\tilde{\mathbb{H}}$ under W^{-1}. This region is analytic, and so by Schwartz reflection ϕ can be extended.

We can normalize such that our point at infinity is fixed under the mapping to 1 on the disk, and expand around its analytic neighborhood as follows:

$$f_t(z) = \alpha z + \beta + \frac{\gamma}{z} + \dots$$

Our map $\phi(z)$ can then be expressed in the usual power series for z near 0, and will be locally injective:

$$\phi(z) = \frac{1}{f_t(\frac{1}{z})} = 0 + a_1 z + a_2 z^2 + \dots$$

By local conformal properties, we know that

$$\phi'(0) = a_1 \neq 0$$

Therefore, we can invert to find that for $\alpha \neq 0$ and $||z|| \gg 1$,

$$f_{_t} = \frac{1}{\phi(\frac{1}{z})} = \frac{1}{a_1\frac{1}{z} + a_2\frac{1}{z^2}\dots} = \alpha z + \beta + \frac{\gamma}{z} + \dots$$

Using the automorphism freedom of the original $\mathbb{H}$, we can take a particular choice of normalization (known as the 'Hydrodynamic Normalization') to set $\alpha = 1, \beta = 0$:

$$f_t(z) = z + \frac{c(t)}{z} + \dots$$

Here, $c(t)$ comes from the original slit and can't be freely adjusted anymore.

We desire a differential equation for f_t. Looking ahead, we'll find one of the form:

$$\dot{f_t}(z) = -\frac{\dot{c(t)}}{f_t(z) - \lambda_t}$$

where the over-dot represents a derivative with respect to the parameter t.

Example To illustrate a constructive map for a simple case, consider the vertical slit.

It can be checked that the map

$$g_t(z) = \sqrt{z^2 + t^2}$$

accomplishes the transform into $\mathbb{H}$. Expanding this map for large z, we find

$$g_t(z) \approx z + \frac{t^2/2}{z} + \dots$$

Therefore, $c(t) = \frac{t^2}{2}$.

Definition A 'hull' is a compact $K \subset \bar{\mathbb{H}}$ such that $\mathbb{H} \backslash K$ is simply connected.

It can be shown that $\forall K$, expansion about infinity yields

$$\begin{aligned} \exists!\, g_k : \mathbb{H}\backslash K &\rightarrow \mathbb{H} \\ g_k(z) &= z + \frac{c(k)}{z} + \dots \end{aligned}$$

$c(k)$ is known as the 'half-plane capacity', or 'hcap.'

Exercise Compute the hcap of the $\tilde{\mathbb{H}}$

Example Given a hull of a half-disk embedded in $\mathbb{H}$, we can map to $\mathbb{H}$ using $z + \frac{1}{z}$. This illustrates that the hcap is 1.

Example Show that $\text{hcap}(rk) = r^2\text{hcap}(k)\,\forall\, r$

First we rescale the set to match the original map, and then we take advantage of our automorphism freedom to rescale the first term in the expansion to be 1 in the hydrodynamic norm, leading to the desired squared scaling of the half-plane capacity.

$$g_{rk}(z) = g_k(\frac{z}{r}) = r g_k(\frac{z}{r}) = z + r\frac{c(k)}{\frac{z}{r}} = z + \frac{r^2 c(k)}{z}$$

Lemma $c(g_1 \circ g_2) = c(g_1) + c(g_2)$

There is a semi-group structure on the compact sets at infinity. The proof is trivial and can be shown as follows:

$$g_1(g_2(z)) = g_2(z) + \frac{c(g_1)}{g_2(z)} + \cdots = z + \frac{c(g_2)}{z} + \frac{c(g_1)}{z + \frac{c(g_2)}{z}}$$

Thus, the $\frac{1}{z}$ term has a coefficient of the sum of each capacity.

There is a natural bijection between the space of compact hulls and hydrodynamically normalized conformal maps.

$$\{hulls\} \leftrightarrow \{conf\ maps\}$$

The correspondence comes through the complement of the hull. Defining the inverse map from the half plane onto the hull space as $f_k = g_k^{-1}$,

$$K = \mathbb{H} \backslash f_k(\mathbb{H})$$

Additional properties are an inverse relation, positivity, monotonicity, and translation invariance:

$$\begin{aligned} c(g^{-1}) &= -c(g) \\ c(k) &\geq 0 \\ c(k) &\leq c(k') \,|\, k \subset k' \\ c(k) &= c(k + x \in \mathbb{R}) \end{aligned}$$

The inequalities are strict for $k = \emptyset$ and $k = k'$ respectively.

We can place naive upper bounds on c by considering a disk of radius r which fully contains the hull k of interest. As we can easily compute $c(D(0,r)) = r^2$, we know that

$$c(k) \leq r^2 \,|\, k \subseteq D(0,r)$$

Exercise Consider the slit-half-plane map given above, f_1. Under the usual random composition of the flipping maps, we have an intuition that the resulting SLE curve will extend off to infinity, gaining both infinite height and infinite width. However, if we repeat the same map f_1 without any reflections, the growth will be capped at some limiting height dependent on the initial angle and linear in the number of compositions. Demonstrate this property.

Another way to define hcap is to consider a Brownian generated curve, and examine the first time it enters boundary of the hull of interest. If we compute the expectation of the height of that intersection point, in the limiting case with appropriate scaling, we will yield hcap.

$$\lim_{y\to\infty} y\,\mathrm{E}[\Im(B^{iy}_{T_{\mathbb{R}\cup K}})] = \mathrm{hcap}(K)$$

Proof We would like to prove the positivity property of hcap.

Consider the map $f : \mathbb{H} \to \mathbb{H}\backslash K$. Assume $f = u+iv$ is continuous up to the real line, or equivalently, $\partial\mathbb{H}\backslash K$ is locally connected (every point has arbitrarily small neighborhoods which are connected, as opposed to the situation encountered with Topologist's Sine Curve).

$\Im(f)$ is supported on some interval defined by K, and is a positive harmonic function whose boundary values we know.

Using the Poisson integral formula,

$$\begin{aligned} \Im(f(z)) &= \frac{1}{\pi}\int_{\mathbb{R}} v(t)\frac{y}{(x-t)^2+y^2}dt \\ \frac{y}{(x-t)^2+y^2} &\sim \Im(\frac{1}{t-z}) \end{aligned}$$

As we have two harmonic functions with the same boundary behavior, they should be the same. However, out at infinity, the left side blows up while the right goes to 0. Instead, we can take

$$\Im(f(z)-z)$$

as this function is the same as the above on the real line, but now it has the right behavior at infinity, making the functions truly identical.

Now, we discard the imaginary restriction. We are allowed to do this, as we have 2 analytic functions with the same imaginary part which differ by at most a real constant, but they agree at infinity and are therefore still identical with

$$\Im(f(z)-z) \to f(z)-z$$

Additionally, we are really just integrating over the compact support of v in the projection of K on $\mathbb{R}$. Multiplying by an extra factor of y and taking the limit where y approaches infinity, we yield, with $z = iy$,

$$\lim_{y\to\infty} y(f(iy)-iy) = \frac{1}{\pi}\int_{sup} v(t)dt \equiv \mathrm{hcap}(k)$$

which is positive definite for non-empty support.

Back to the Loewner equation, given a curve, we first need to prove that the complement is continuous. A continuous, increasing function that starts at 0 (in our case) always admits a parameterization, and so we know that we can re-parameterize it such that $\text{hcap}(\gamma[0,t]) = 2t$. With this imposed, the Loewner equation becomes

$$\frac{d}{dt}g_t(z) = \frac{2}{g_t(z) - \lambda_t}$$

where $\lambda_t = g_t(\gamma(t))$.