Lecture 6: Continuation of proof of Loewner Differential Equation

Recall from *April 6*:

Let $\gamma(t)$ be a curve in $\mathbb{H}$, starting from the origin. For simplicity, we assume $\gamma(t)$ is a simple curve. Let $g_t(z)$ be the hydrodynamically normalized conformal map from $H_t := \mathbb{H}\backslash\gamma([0,t])$ to $\mathbb{H}$.

Theorem 1 *$\lambda(t) = g_t(\gamma(t))$ exists, is continuous (in t), $t \mapsto g_t(z)$ is differentiable, and we have the Loewner Differential Equation:*

$$\frac{d}{dt}g_t(z) = \frac{2}{g_t(z) - \lambda(t)}$$

Proof: Last class, we showed that $\lambda(t)$ is continuous in t. Let $s > t$ and let $g_{t,s} = g_s \circ g_t^{-1}$ be the conformal map from $g_t(H_s)$ to $\mathbb{H}$. Note that $g_{t,s}$ converges to the identity in the Carathéodory topology.

Define $\psi := \psi_{t,s} = g_{t,s}^{-1}$. $\psi : \mathbb{H} \to \mathbb{H}\backslash g_t(\gamma([t,s]))$. Note that ψ is already hydrodynamically normalized: $\psi(z) = z + \frac{2(t-s)}{z} + \ldots$. Recall from *April 1*,

$$\mathrm{Im}(\psi(z) - z) = \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Im}\frac{1}{t-z}\mathrm{Im}\psi(t)dt \qquad \text{and}$$

$$\psi(z) - z = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{t-z}\mathrm{Im}\psi(t)dt$$

Hence, letting $z = iy$ and letting $y \to \infty$, we get

$$-2(t-s) = \frac{1}{\pi}\int_{\mathrm{supp}(\psi)} \mathrm{Im}\psi(t)dt$$

Let $I_{s,t} = \mathrm{supp}(\psi)$ and observe that $|I_{s,t}| \to 0$ as $s \to t$. Let a be any point in $I_{s,t}$ and note that $a \to \lambda(t)$ as $s \to t$. We have that

$$\begin{aligned}
\frac{\psi(z) - z}{s-t} &= \frac{1}{\pi}\frac{1}{s-t}\int_{I_{s,t}} \frac{1}{t-z}\mathrm{Im}\psi(t)dt \\
&= \frac{1}{\pi}\frac{1}{s-t}\int_{I_{s,t}} \left(\frac{1}{a-z} + \mathcal{O}(|I_{s,t}|)\right)\mathrm{Im}\psi(t)dt \\
&= 2\left(\frac{1}{a-z} + \mathcal{O}(|I_{s,t}|)\right)
\end{aligned}$$

Therefore,

$$\lim_{s\to t} \frac{\psi(z)-z}{s-t} = \frac{2}{\lambda_t - z}.$$

By a simple change of variables (let $z = g_s(w)$) we get the Loewner Differential Equation.

Example 1 *Let $\tilde{\gamma}(t) = it$. Let $f_t(z) = \sqrt{z^2+t^2}$ be a conformal map from H_t onto $\mathbb{H}$.*

$$f_t(z) = z(1 + \tfrac{t^2/2}{z^2} + ...) = z + \tfrac{t^2/2}{z} + ...,$$

so in order to parameterize our curve so that the coefficient on z^{-1} is $2t$, we must take $\gamma(t) = \tilde{\gamma}(2\sqrt{t}) = i2\sqrt{t}$. Thus,

$$g_t(z) = \sqrt{z^2 + 4t}$$

Taking derivatives, then, we get that

$$\frac{d}{dt} g_t(z) = \frac{1}{2g_t(z)} \cdot 4 = \frac{2}{g_t(z) - 0}$$

I.e., in this case, $\lambda(t) \equiv 0$.

Exercise 1 *Recall that if the line segment in **Figure 1** (call it γ) is of unit length, then a conformal map from $\mathbb{H}$ onto $\mathbb{H}\backslash\gamma$ is $f = c_\alpha(z-\alpha)^\alpha(z-(\alpha-1))^{1-\alpha}$.*

Show that $\lambda(t) = \dfrac{2(1-2\alpha)}{\sqrt{\alpha(\alpha-1)}}\sqrt{t}$.

Exercise 2 *Show that $\lambda(t) = 3\sqrt{2}\sqrt{1-t}$ gives **Figure 2**.*

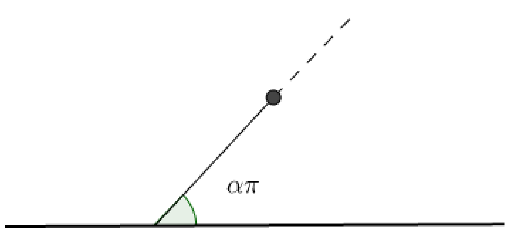

Figure 1: Exercise 1

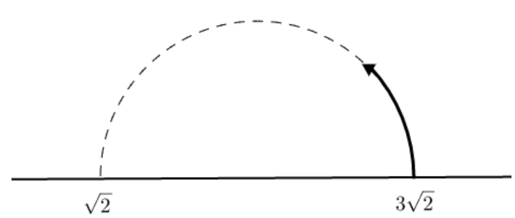

Figure 2: Exercise 2