SLE	**Lecture 9, 10**
MATH 583 Spring 2016	**April 15-18, 2016**

Lecture 9: Going from $\lambda(t)$ to $\gamma(t)$

April 15:

Example 1 *Let*

$$\lambda(t) = \begin{cases} 3\sqrt{2} & 0 \le t \le T \\ 3\sqrt{2}\sqrt{T+1-t} & T \le t \le T+1 \\ 0 & T+1 \le t < \infty \end{cases}$$

for some $T < \infty$. What $\gamma(t)$ does this give?

Here is a rough sketch of what $\gamma(t)$ will look like:

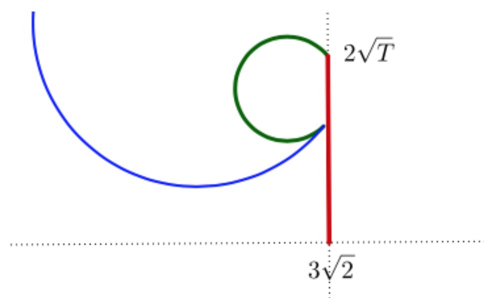

Figure 1: Sketch of **Example 1**

We can see this by considering that up until time T, γ is the vertical line starting from $3\sqrt{2}$ of height $2\sqrt{T}$. This is the **red** portion of the sketch. We then consider the image of $\gamma([T, T+1])$ under $g_T(z)$ and realize it is the semicircle with endpoints $\sqrt{2}$ and $3\sqrt{2}$. So if $\sqrt{2} > 3\sqrt{2} - 2\sqrt{T}$, then the image of this semicircle under $g_T^{-1}(z)$ is a circular arc that intersects our red segment. This is shown by the **green** portion of the sketch. Lastly, we consider the image of $\gamma([T+1, \infty))$ under the map $g_{T+1}(z)$. This is the vertical ray starting at 0. The image of this ray under $g_{T+1}^{-1}(z)$ is an arc that bisects the angle between the red and green portions of the picture and then goes off toward infinity. This is (roughly) indicated by the **blue** portion of the sketch.

In previous classes, we saw how to obtain a driving function $\lambda(t)$ from a curve $\gamma(t)$. Now, conversely, suppose that we are given a continuous function $\lambda(t) : [0, T) \to \mathbb{R}$. Consider the equation:

$$\frac{d}{dt} z(t) = \frac{2}{z(t) - \lambda(t)} \tag{1}$$

We have the following lemma:

Lemma 1 *Let $z_0 \in \mathbb{C}\backslash\{\lambda(0)\}$. There exists a maximal nonempty interval $[0, T_{z_0})$ on which the initial value problem (1) with $z(0) = z_0$ has a unique solution. If $T_{z_0} < T$, then $z(t) - \lambda(t) \to 0$ as $t \nearrow T_{z_0}$.*

Moreover, if we set $K_t := \{w \in \mathbb{H} : T_w \leq t\}$, then $H_t = \mathbb{H}\backslash K_t$ is open, simply connected, and $g_t : H_t \to \mathbb{H}$ is analytic, one-to-one, onto, and $g_t(w) = w + \frac{2t}{w} + \mathcal{O}(\frac{1}{w^2})$ for w near ∞, where $g_t(w)$ is the solution to the differential equation with initial condition w.

Proof: (Sketch)
We will use the Picard-Lindelöf iteration scheme. Let $\dot{z}(t) = f(w,t)$. Here we assume f is L-bi-Lipschitz in w (i.e. $|f(w_1) - f(w_2)| \leq L|w_1 - w_2|$) and continuous in t. (**Note:** In our case, f is not Lipschitz everywhere: f blows up when $z(t)$ is near $\lambda(t)$. We can account for this by writing

$$f(w,t) = \begin{cases} \frac{1}{z(t)-\lambda(t)} & y \geq \epsilon \\ \frac{1}{x+i\epsilon-\lambda(t)} & y \leq \epsilon \end{cases}$$

and adjusting the proof accordingly by taking limits, etc. The details are left to the reader.)

Inductively define $z_n(t)$:

$$z_0(t) = z_0$$
$$z_{n+1}(t) = z_0 + \int_0^t f(z_n(s), s)ds$$

Then

$$\begin{aligned} |z_{n+1}(t) - z_n(t)| &\leq \int_0^t |f(z_n(s), s) - f(z_{n-1}(s), s)|ds \\ &\leq L \int_0^t |z_n(s) - z_{n-1}(s)|ds \\ &\leq L \cdot ||z_n - z_{n-1}||_{\infty,[0,t]} \cdot t \end{aligned}$$

Therefore, for $\tau < \infty$,

$$||z_{n+1} - z_n||_{\infty,[0,\tau]} \leq L\tau \cdot ||z_n - z_{n-1}||_{\infty,[0,\tau]}$$

So if $L\tau < 1$, $\{z_n\}_{n\geq 0}$ converges in the uniform norm to a continuous function which satisfies the integral equation $z(t) = z_0 + \int_0^t f(z(s), s)ds$, and hence, the differential equation $\dot{z} = f(z(t), t)$ with initial condition $z(0) = z_0$.

(Continued on April 18:)
To see that $g_t(w)$ is the hydrodynamically normalized conformal map from H_t onto $\mathbb{H}$, consider that it is analytic as it is the uniform limit of analytic functions. It is easy to see that it is

injective and surjective by "flowing back," i.e., consider the map $s \mapsto z(T-s)$. It is also clear that g_t does not map into the lower half plane, as once a particle in the flow hits the real line, it must remain in the real line as the ODE is now real.

Lastly, we must show that $g_t(z) = z + \frac{2t}{z} + \mathcal{O}(\frac{1}{z^2})$ for z near ∞. Observe that near ∞, g_t is well-defined, one-to-one, and analytic (∞ is a removable singularity), and $g_t(\infty) = (\infty)$. We write

$$g_t(z) = a_1(t)z + a_0(t) + a_{-1}(t)\frac{1}{z} + ...$$

By Cauchy's integral formula, $\forall R < \infty$,

$$a_n(t) = \frac{1}{2\pi i}\int_{\partial D_R(0)} g_t(z)\frac{1}{z^{n+1}}dz$$

Observe that

$$\begin{aligned} |\dot{a}_n(t)| &= |\frac{1}{2\pi i}\int_{\partial D_R(0)} \frac{2}{g_t(z)-\lambda(t)}\frac{1}{z^{n+1}}dz| \\ &\leq R \max_{\partial D_R(0)} \frac{2}{|g_t-\lambda_t|R^{n+1}} \\ &\xrightarrow{R\to\infty} 0 \qquad \text{if } n = 0, 1 \end{aligned}$$

Hence, a_0 and a_1 are constants. For $n = -1$, we have

$$\begin{aligned} \dot{a}_{-1}(t) &= \frac{1}{2\pi i}\int_{\partial D_R(0)} \frac{2}{z-\lambda(t)} + \underbrace{\left(\frac{2}{g_t(z)-\lambda(t)} - \frac{2}{z-\lambda(t)}\right)}_{\mathcal{O}(\frac{1}{R^2})} dz \\ &\xrightarrow{R\to\infty} 2 \end{aligned}$$

Thus, $\dot{a}_{-1}(t) \equiv 2$. Therefore, observing that $a_1(0) = 1$ and $a_0(0) = 0 = a_{-1}(0)$ since $g_0(z) = z$, we have that $a_1(t) = 1$, $a_0(t) = 0$ and $a_{-1}(t) = 2t$. $\square$

Corollary 1 $K_t \neq \emptyset$ *for all* $t > 0$.

Proof: $g_t : \mathbb{H}\backslash K_t \to \mathbb{H}$ is the unique normalized conformal map. If $K_t = \emptyset$, then $g_t(z) = z$, which is a contradiction as the coefficient of z^{-1} in the expansion of g_t should be $2t$. $\square$

Lemma 2 $K_t \subset D_{4R}(0)$, *where* $R = \max\{\sqrt{t}, ||\lambda||_{\infty,[0,t]}\}$.

Proof: Fix $|z| \geq 4R$.

Claim: $|g_s(z) - z| \leq R$ for $0 \leq s \leq t$.

The claim implies $|g_s(z)| \geq 4R - R = 3R$ and $|g_s(z) - \lambda_s| \geq 3R - R = 2R$. Thus, $z \notin K_t$.

Suppose the claim is not true: then there is a first time σ for which $|g_\sigma(z) - z| = R$. So for all $s \leq \sigma$,

$$|\dot{g}_s(z)| = \left|\frac{2}{g_s(z) - \lambda_s}\right| \leq \frac{1}{R} \Longrightarrow |g_s(z) - z| \leq \sigma \frac{1}{R}$$

Letting $s = \sigma$, we have $\sigma \geq R^2$. Contradiction. □.

Theorem 1

- *If γ is a simple curve in $\mathbb{H} \cup \{\gamma(0)\}$, then $K_t = \gamma([0,t])$. Otherwise, $K_t = (\text{comp}_\infty \mathbb{H} \backslash \gamma([0,t]))^c$ (where comp_∞ indicates the unbounded component).*
- *An increasing collection $\{K_t\}_{t \geq 0}$ with $\text{hcap} K_t = 2t$ is the set of hulls for a continuous $\lambda(t)$ if and only if $K_{t+s} \backslash K_t$ can be separated from ∞ by a curve $\sigma \subset \mathbb{H} \backslash K_t$ with small diameter.*

Proof: Exercise.