SLE	**Lecture 8**
MATH 583 Spring 2016	**April 13, 2016**

Lecture 8: Proof of Loewner's Lemma and Properties of Driving Functions

Recall from *April 6*:

If K is a connected hull of $\mathbb{H}$ (i.e., $K \subset \overline{\mathbb{H}}$ and $\mathbb{H}\backslash K$ is simply connected), and $g_K(z)$ is the hydrodynamically normalized conformal map from $\mathbb{H}\backslash K$ to $\mathbb{H}$, then we have the following lemma:

Lemma 1 ***(Loewner's Lemma)*** $0 < g'_K(x) \leq 1$ *on* $\mathbb{R}\backslash\overline{K}$.

We now give the proof of this lemma:

Proof: Fix $x \in \mathbb{R}\backslash\overline{K}$. Let $I \subset \mathbb{R}$ be an open interval such that $x \in I$. Recall from *April 11* that

$$|I| \sim \pi y \cdot \omega(iy, I, \mathbb{H})$$

where $\omega(iy, I, \mathbb{H})$ is the harmonic measure from iy of I with respect to the domain $\mathbb{H}$ and "$\sim$" denotes asymptotic equivalence as $y \to \infty$. Then we get that

$$\begin{aligned}
|I| &\sim \pi y \cdot \omega(iy, I, \mathbb{H}) \\
&\geq \pi y \cdot \omega(iy, I, \mathbb{H}\backslash K) && \text{(use the maximum priniciple)} \\
&= \pi y \cdot \omega(g_K(iy), g_K(I), \mathbb{H}) && \text{(conformal invariance of harmonic measure)} \\
&\sim \pi y \cdot \omega(iy, g_K(I), \mathbb{H}) \\
&\sim |g_K(I)|
\end{aligned}$$

So, letting $y \to \infty$, we have $|g_K(I)| \leq |I|$. Letting $|I| \to 0$, we get $|g'_K(x)| \leq 1$. Noting that $g'_K(x) > 0$ (*exercise*), the result follows. $\square$

We now move on to properties of driving functions. We have the following lemma, partially left as an exercise:

Lemma 2 *If* $\gamma(t) \in \mathbb{H} \cup \{\gamma(0)\}$ *is driven by* $\lambda(t)$ *and assuming that* $\text{hcap}(\gamma([0, t]) = 2t)$*, then*

1. (translation) $\gamma + x$ *is driven by* $\lambda_t + x$
2. (concatenation) $g_{t_0}(\gamma(t))|_{t \geq t_0}$ *is driven by* $\lambda(t + t_0)$

3. (reflection) *If $\hat{\gamma}$ is the reflection of γ in $i\mathbb{R}$ then $\hat{\gamma}$ is driven by $-\lambda(t)$*

4. (scaling) *For $r > 0$, $r\gamma(t)$ is driven by $r\lambda(t)$*

Proof:

1. If $\hat{\gamma(t)} = \gamma(t) + x$, then $\hat{g}_t(z) = g(z - x) + x$, so $\hat{g}_t(\hat{\gamma}(t)) = \lambda_t + x$.
2. This is clear when you consider that $g_t \circ g_{t_0}^{-1}(z)$ is the hyrodynamically normalized conformal map that maps $\mathbb{H} \backslash g_{t_0}(\gamma([0, t]))$ to $\mathbb{H}$.
3. $\hat{\gamma(t)} = -\overline{\gamma(t)}$, so $\hat{g}_t(z) = -\overline{g(-\overline{z})}$. Thus, $\hat{g}_t(\hat{\gamma}(t)) = -\lambda_t$ as λ_t is real-valued.
4. *Exercise.*

Example 1 Let $\gamma = \{te^{i\alpha} : t > 0\}$ be a ray from the origin. Let $\lambda(t)$ be the driving function of γ. Note that $r\gamma = \gamma$ for $r > 0$. So by the lemma, $\lambda(t) = r\lambda(\frac{t}{r^2})$ for all t, r. Setting $r = \sqrt{t}$, we get $\lambda(t) = \sqrt{t} \cdot \lambda(1)$.

Exercise: find an expression for $\lambda(1)$ in terms of α.

Recall that we defined SLE_κ by setting the driving function to be $\sqrt{\kappa}B(t)$ where $B(t)$ is a standard Brownian motion. Brownian motion has the property (known as "Brownian scaling") that for $r > 0$, $\tilde{B}(t) = rB(\frac{t}{r^2})$ is another standard Brownian motion, i.e. $\tilde{B}(t) \overset{d}{=} B(t)$. Thus, by the lemma (part 4), solutions to Loewner's Differential Equation driven by Brownian motion will be scale invariant.