Brownian Motion: A Crash Course

Here we present some useful definitions and review some historical results and relevant facts of the extensively studied topic of Brownian motion. We begin with an intuitive definition of Brownian motion that builds on the notion of a simple random walk (SRW).

Let $S_n = X_1 + X_2 + \cdots + X_n$ be a one dimensional simple random walk where the X_i's are i.i.d real random variables and $\mathbb{P}(X_i = \pm 1) = \frac{1}{2}$. The SRW can be thought of as flipping an unbiased coin, where each time heads comes up we get the value 1 and each time tails comes up we get the value -1. Thus, if we start at the origin and interpolate linearly between points, after n flips we will have a simple random path defined on $[0, n]$ (shown in Figure 1).

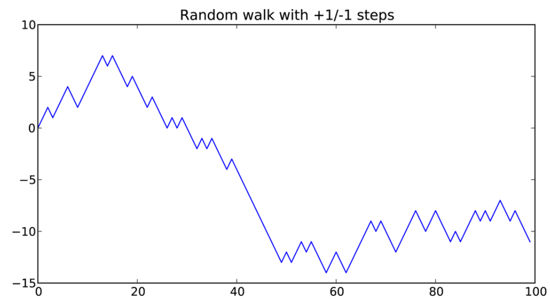

Figure 1: Graphical depiction of a simple random walk. The x-axis is indexed by the number of flips n whereas the y-axis denotes the sum S_n.

Now a question that naturally arises is what is the longterm behavior of S_n on average? The law of large numbers states that the sequence $\frac{S_n}{n} \overset{n\to\infty}{\longrightarrow} 0$. Since, this is a fairly intuitive result and doesn't give us much to work with, we try again except this time, scale by $\sqrt{n}$. It turns out that $\frac{S_n}{\sqrt{n}}$ converges to the standard normal distribution as $n \to \infty$, i.e.

$$\mathbb{P}\left(a < \frac{S_n}{\sqrt{n}} < b\right) \overset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} \, dx.$$

As an exercise, it is suggested the reader prove the previous claim (hint: consider the characteristic function $\mathbb{E}[e^{it\frac{S_n}{\sqrt{n}}}]$).

Coming back to the picture of the SRW we consider the case for N very large. From the previous result we now know that if we look at the value $\frac{S_N}{\sqrt{N}}$ we can say almost surely that this number is within a small range that is bounded between -3 and 3 (with probability approximately equal to 0.9987 guaranteed by the standard normal distribution). If we then scale the whole SRW picture by $\sqrt{N}$ we arrive at Brownian motion. This surprising result is known as Donsker's invariance principle which informally states that the random function generated by linearly interpolating the integer points of the SRW converges to Brownian motion when properly scaled to live on the domain $[0, 1]$.

We now present the formal definition of Brownian motion.

Definition. *A standard (1-d) Brownian motion is a stochastic process* $B_t : \Omega \to \mathbb{R}$ *on the probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *such that* *i.* $B_0 = 0$

ii. *If* $t_1 < t2 < \cdots < t_n$ *then* $B_{t_1}, B_{t_2 - t_1}, \cdots, B_{t_n - t_{n-1}}$ *are independent.*

iii. $B_t - B_s \sim \mathcal{N}(0, |t-s|)$

iv. $t \mapsto B_t(\omega)$ *is continuous with probability 1.*

Constructing Brownian motion

Now that we have developed some intuition for what Brownian motion is, we set out to understand how one constructs such an object. We previously mentioned Donsker's invariance principle, wherein one obtains Brownian motion as the limiting behavior in distribution of scaled copies of a random walk, which we shall return to later. Another possible construction involves taking i.i.d. random variables following the standard normal distribution and treating them as coefficients of a random Fourier series. We shall, instead first, give a brief summary of Levy's algorithmic construction of Brownian motion.

The idea is as follows: we construct a sequence of piecewise linear continuous functions on the set of dyadic points $\mathcal{D} = \bigcup_0^\infty \mathcal{D}_n$(where $\mathcal{D}_n = \{\frac{k}{2^n} : 0 \leq k \leq 2^n\}$) that converges uniformly to standard Brownian motion B_t. Let $Z_t : t \in \mathcal{D}$ denote a collection of i.i.d. standard Normal random variables and to begin we set $B_0 := 0,\ B_1 := Z_1$ so that B_t is defined on $\mathcal{D}_0$. We continue, by setting $B_{\frac{1}{2}} := \frac{1}{2}B_1 + \frac{1}{2}Z_{\frac{1}{2}}$ so that B_t is now defined on D_1. We continue the process inductively and put

$$B_t = \frac{B_{t-2^{-n}} + B_{t+2^{-n}}}{2} + \frac{Z_t}{2^{(n+1)/2}}$$

which is defined on $\mathcal{D}_n$. We then define the function

$$F_0(t) = \begin{cases} Z_1,\ t = 1 \\ 0,\ t = 0 \\ \text{linear between } t = 0 \text{ and } t = 1 \end{cases}$$

$$F_{n\geq 1}(t) = \begin{cases} 2^{-(n+1)/2} Z_t,\ t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1} \\ 0,\ t \in \mathcal{D}_{n-1} \\ \text{linear between consecutive points in } \mathcal{D}_n. \end{cases}$$

With some careful analysis it can be shown that

$$B_t = \sum_{i=0}^{n} F_i(t) = \sum_{i=0}^{\infty} F_i(t).$$

For a more thorough treatment we direct the reader to Peres & Morters' text *Brownian Motion.*

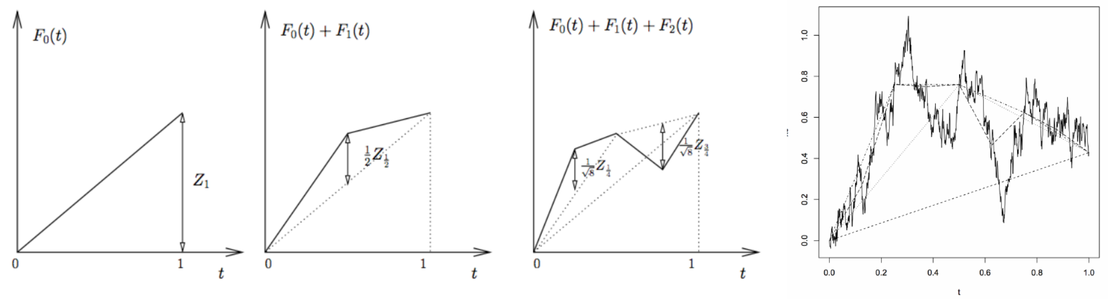

Figure 2: The first three images show the first three iterations of the function F_n. The final image shows the function F_n after several iterations.

We now present two useful Lemma's.

Lemma 1. *X, Y are i.i.d Gaussian random variables iff $X \pm Y$ are i.i.d. Gaussian random variables.*

Proof. The proof is left as an exercise.

Lemma 2. *Let B_t denote a standard Brownian motion and assume $r > 0$. Then the process $X_t = \frac{1}{r} B_{r^2 t}$ is also a standard Brownian motion.*

Proof. All we must show is that the increments of X_t are independent normal variables, since continuity, independence and stationarity remain unchanged under scaling. Let $t > s$, apply Lemma 1 and observe the following:

$$X_t - X_s = \frac{1}{r^2}(B_{r^2 t} - B_{r^2 s}) = \mathcal{N}(0, \frac{1}{r^2}(r^2 t - r^2 s) = \mathcal{N}(0, t - s).$$

Hence, the increments of X_t are normally distributed as desired and X_t is a standard Brownian motion. ■

We continue by presenting some known facts without proof:

- As a rough estimate $B_t \approx \sqrt{t}$ ($B_t \overset{d}{=} \sqrt{t} B_1$).
- (Modulus of Continuity). For t fixed, $C > \sqrt{2}$, and for all h small enough we have the following:

$$|B_{t+h} - B_t| \leq C \sqrt{h \log\left(\frac{1}{h}\right)}.$$

- (Law of the iterated logarithm). With probability 1 we have the following:

$$\lim_{h \to 0} \frac{B_h}{\sqrt{h \log \log\left(\frac{1}{h}\right)}} = \sqrt{2}.$$

Returning to the SRW we now give a brief sketch of the proof of Donsker's invariance principle, which is also known as the functional central limit theorem. Recall the random walk $S_n = X_1 + X_2 + \cdots + X_n$ where each consecutive point of the walk is linearly interpolated with its two nearest neighbors.

Theorem (Donsker's Invariance Principle). *Put* $S_n^*(t) = \frac{S_{nt}}{\sqrt{nt}}$ *for* $t \in [0,1]$. *Then on the space* $\mathcal{C}[0,1]$ *of continuous functions on the unit interval with the metric induced by the sup-norm, the sequence of random functions* $S_n^*(t)$ *converges in distribution to a standard Brownian motion* B_t *with* $t \in [0,1]$.

For a full treatment of the proof we again direct the reader to Peres & Morters' text *Brownian Motion*. Here we present a very rough sketch. The idea is to embed the sequence of random variables $X_1, X_2, \cdots, X_n$ in the same probability space as the Brownian motion such that $S_n^*(t)$ is almost surely close to a scaling of Brownian motion.

Thus, the first step is to consider a standard Brownian motion B_t and come up with a clever way to think of it as a SRW. One way to do this is to consider the times T_n at which B_t intersects the horizontal integer lines $\{B_t = n : n \in \mathbb{Z}\}$. Formally we set

$$T_1 := \inf\{t : |B_t| = 1\},\ T_{n+1} := \inf\{t > T_n : |B_t - B_{T_n}| = 1\}$$

and we call T_n a *stopping time*.

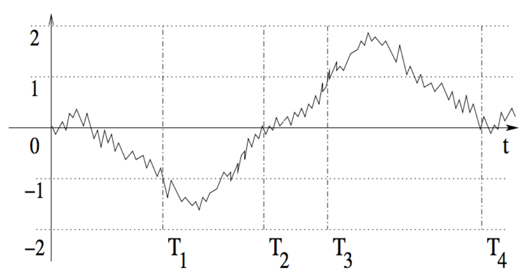

Figure 3: Example of defining stopping times for a SRW.

From here we proceed by invoking a Skorokhod embedding, which informally, states the existence of a stopping time T such that B_T follows the law of a random variable X for which $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] < \infty$, and $\mathbb{E}[T] = \mathbb{E}[X^2]$.

Now let X be a real random variable with mean 0 and variance 1 and define T_1 to be such that $\mathbb{E}[T_1] = 1$ and $B_{T_1} = X$ in distribution, using the Skorokhod embedding. Similarly, we then define T_2' to be such that $\mathbb{E}[T_2'] = 1$ and $B_{T_2} = X$ in distribution, so that $T_2 = T_1 + T_2'$ and $\mathbb{E}[T_2] = 2$. Continuing in this fashion, we inductively define a sequence of stopping times $T_1 < T_2 < \cdots < T_n$ such that $S_n = B_{T_n}$, i.e. the Brownian motion with stopping times T_n has the same distribution of the SRW described by S_n.

Sweeping the technical details under the rug, now that we have found a usable embedding that satisfies the correct properties, the conclusion is that, after rescaling our Brownian motion, the difference $\left|\frac{B_{nt}}{\sqrt{nt}} - S_n^*\right|$ becomes negligible as $n \to \infty$.