April 29
Recall from *April 27*: consider a Brownian motion starting from $x \in (a, b)$, define a stopping time $T = \inf\{t > 0, B_t^x = a \text{ or } b\}$. We showed

$$\mathbb{P}(B_T^x = a) = \frac{b - x}{b - a} \tag{1}$$

by using the optional stopping theorem. **Caution**: if instead we define the stopping time to be $T = \inf\{t > 0, B_t^x = a\}$, we know $T < \infty$ a.e. Clearly

$$E(B_T^x) = a \neq x = E(B_0^x).$$

Why doesn't the optional stopping theorem apply here? (Hint: $ET = \infty$.)

Firstly let's prepare ourselves with some knowledge of martingales.

Definition 1. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let X be a random variable and $\mathcal{G}$ be a sub-algebra of $\mathcal{F}$. A random variable Y is called (a version of) the **conditional expectation** of X with respect to $\mathcal{G}$ (denoted as $E(X|\mathcal{G})$) if Y is $\mathcal{G}$-measurable, and

$$\int_A Y d\mathbb{P} = \int_A X d\mathbb{P}, \quad \forall A \in \mathcal{G}.$$

Remark. If $X \in L^2$, one can show that $E(X|\mathcal{G})$ is the orthogonal projection of X on the space $L^2(\Omega, \mathcal{G}, \mathbb{P}) \hookrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2. A **martingale** with respect to filtration $\{\mathcal{F}_t\}$ is a stochastic process M_t such that

- $E|M_t| < \infty \quad \forall t$.
- $E(M_t|\mathcal{F}_s) = M_s \quad \forall 0 \leq s < t$.

Remark. Intuitively, a martingale is understood as a fair game.

Example 3.

(1) The simple random walk S_n and the Brownian motion B_t are discrete/continuous-time martingage. (The natural filtration of Brownian motion is $\mathcal{F}_t := \sigma(B_t) \in \mathcal{F}$.)

(2) Let $f : x \in [0, 1) \to \mathbb{R}$ be an arbitrary function. Think of $x \in [0, 1)$ by its binary expansion $x = 0.\alpha_1\alpha_2\cdots$, where $\alpha_i = 0$ or 1 depending whether the i-th coin toss gives head or tail. Consider the filtration

$$\mathcal{F}_n = \left\{\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)\right\}_j,$$

and

$$M_n(x) = \frac{1}{|I_n(x)|} \int_{I_n(x)} f(t)dt,$$

where $I_n(x) = [j/2^n, (j+1)/2^n)$ *is the dyadic interval containing* x. *Check that this is a martingale*

(3) $B_t^2 - t$ *is a martingale, since*

$$\begin{aligned} E(B_t^2 - t|\mathcal{F}_s) &= E\left((B_t - B_s)^2 + B_s^2 + 2B_s(B_t - B_s) - t|\mathcal{F}_s\right) \\ &= E(B_t - B_s)^2 + B_s^2 + 2B_s \cdot E(B_t - B_s|\mathcal{F}_s) - t \\ &= B_s^2 - s. \end{aligned}$$

Now we go back to the topic of SLE. Recall the Loewner differential equation

$$\dot{z}_t = \frac{2}{z_t - \lambda_t}, \quad \text{where } \lambda_t = \sqrt{\kappa} B_t.$$

K_t is the killing set at time t.

Exercise 4. *Prove the following statements about* K_t.

- $\operatorname{diam} K_t \to \infty$
- $\max\{\operatorname{Im} z : z \in K_t\} \to \infty$
- $\max\{\operatorname{Re} z : z \in K_t\} \to \infty$

Proof.

- Assume for contradiction that $K_t \subset D(0, R)$, then

$$\operatorname{hcap}(K_t) \leq \operatorname{hcap}(D(0, R)) = R^2$$

is bounded. However $\operatorname{hcap}(K_t) = 2t \to \infty$ as $t \to \infty$. Note that this proof actually works for Loewner evolutions with any driving function λ_t.

- Let $X_t = \max \operatorname{Im} K_t$. By the scaling invariant of the Loewner evolution

$$X_t \overset{d}{=} \max \operatorname{Im} rK_{t/r^2} = rX_{t/r^2},$$

thus $X_t \to \infty$ if X_t starts off to be (strictly) positive. This happens for sure since $K_t \neq \emptyset$.

- A priori the above argument does not work for $Y_t = \max \operatorname{Re} K_t$, since Y_t could start off negative. However, let A_- denote the event that K_t initially sits in the left half, then by Blumenthal 0-1 law we know $\mathbb{P}(A_-) = 0$ or 1. By symmetry $\mathbb{P}(A_-) = 1$ is not possible, hence $\mathbb{P}(A_-) = 0$.

□

Roughly speaking, for $\kappa \le 4$, SLE_κ are simple curves; for $4 < \kappa < 8$ we start to have self intersecting curves; and for $\kappa \ge 8$ there are even space-filling curves. We are going to prove certain results within this grand scheme.

Theorem 5. *If $\kappa > 4$, SLE_κ does not generate simple curves a.e.*

Proof. Without loss of generality assume $\gamma(0) = 0$. Assume γ is a simple curve, then any z not on γ will never be killed. In particular for any $x \in \mathbb{R} \setminus \{0\}$, $g_t(x)$ is defined for all t. For $x > 0$, let $X_t = g_t(x) - \lambda_t$. Then X_t is defined and is strictly positive for all t. However, we will show this is not the case if $\kappa > 4$.

Fix $0 < a < x < b < \infty$ (think of a close to 0 and b close to infinity), it is a simple exercise to show $\mathbb{P}(X_t \in (a, b)\ \forall t) = 0$. Define the stopping time $T = \inf\{t > 0 : X_t \notin (a, b)\}$, what is the probability that $X_T = a$? We want to use optional sampling similar to the example (1). Note that X_t is not a martingale, but on the other hand, is it possible to find a function f such that $f(X_t)$ is a martingale? To do this we need to introduce Itô integral and Itô formula. The exact construction and definition will be stated after the proof, but for now let's first see how stochastic calculus is applied in this specific example.

Combining the Loewner differential equation $\dot{g}_t(x) = 2/X_t$ and $\lambda_t = \sqrt{\kappa} B_t$ we have

$$dX_t = \frac{2}{X_t} dt - \sqrt{\kappa} dB_t. \tag{2}$$

Itô formula says

$$df(X_t) = f'(X_t) dX_t + \frac{1}{2} f''(X_t)(dX_t)^2. \tag{3}$$

Since $B_t \approx \sqrt{t}$, intuitively $dt \cdot dB_t \approx (dt)^{3/2}$ and $(dB_t)^2 \approx dt$. Dropping the higher order terms, we get

$$(dX_t)^2 = \left(\frac{2}{X_t} dt - \sqrt{\kappa} dB_t \right)^2 = \kappa dt.$$

Hence (3) becomes

$$\begin{aligned} df(X_t) &= f'(X_t) \left(\frac{2}{X_t} dt - \sqrt{\kappa} dB_t \right) + \frac{1}{2} f''(X_t) \kappa dt \\ &= \left(\frac{2}{X_t} f'(X_t) + \frac{\kappa}{2} f''(X_t) \right) dt - \sqrt{\kappa} f'(X_t) dB_t. \end{aligned}$$

$f(X_t)$ is a martingale if and only if the drifting term vanishes, i.e.

$$\frac{2}{X_t} f'(X_t) + \frac{\kappa}{2} f''(X_t) = 0. \tag{4}$$

One can easily check

$$f(X_t) = c X_t^{1 - \frac{4}{\kappa}}$$

is a solution to (4), and thus such $f(X_t)$ is a martingale. By the optional stopping theorem

$$E(f(X_T)) = E(f(X_0)) = f(x).$$

(The stopping time T is not necessarily bounded, however we may consider bounded stopping time $T \wedge \tau$ and let $\tau \to \infty$. We omit the details here.) Therefore simple calculations show that

$$\mathbb{P}\left(X_T = a\right) = \mathbb{P}\left(f(X_T) = f(a)\right) = \frac{b^{1-\frac{4}{\kappa}} - x^{1-\frac{4}{\kappa}}}{b^{1-\frac{4}{\kappa}} - a^{1-\frac{4}{\kappa}}}.$$

If $\kappa > 4$, let $a \to 0$ and $b \to \infty$, we obtain $\mathbb{P}(X_T = 0) = 1$. □

The formula (2) is understood in terms of integration:

$$X_t - X_0 = \int_0^t \frac{2}{X_s} ds + \int_0^t \sqrt{\kappa} dB_s.$$

In general, how do we make sense of stochastic integral of the form $\int_a^b f(t,\omega)dB_t$? The usual Riemann-Stieltjes integral does not work here, because B_t does not have bounded total variation. Moreover, stochastic integral exhibits some interesting phenomenon which we do not see in deterministic integral, and we are going to illustrate it in the following example.

Example 6. *We want to define $\int_a^b B_s dB_s$. Given a partition of $[a,b]$, $a = t_0 < t_1 < \cdots < t_n = b$, there are at least two ways to define the Riemann sum:*

$$\sum_{k=1}^{n} B_{t_{k-1}} \left(B_{t_k} - B_{t_{k-1}}\right); \tag{5}$$

$$\sum_{k=1}^{n} B_{t_k} \left(B_{t_k} - B_{t_{k-1}}\right).$$

Using properties of the Brownian motion, more precisely using the fact that $B_{t_k} - B_{t_{k-1}} \sim \mathcal{N}(0, t_k - t_{k-1})$ and is independent of $B_{t_{k-1}}$, one can easily verify the first definition has expectation 0, and the second has expectation $b-a$. Thus taking the left endpoints as sample points, as in the first definition (5), is what we need and is in fact how Itô defined stochastic integral. Although Itô integral is often used in mathematics, an alternative definition is given by computing the Riemann sum using the midpoints as sample points. Integral defined as such is called Stratonovich integral, and is widely used in finance.

In general, let f be a square-integrable function in the product space:

$$\mathcal{A} = \{f(t,\omega) \in \mathcal{B} \otimes \mathcal{F} : \int_\Omega \int_a^b |f|^2 dt d\mathbb{P} < \infty\},$$

here $\mathcal{B}$ and $\mathcal{F}$ are the $\sigma-$algebra of $[a,b]$ and Ω respectively. Assume we can approximate f by simple functions of the form

$$g(t,\omega) = \sum_k g_k(\omega)\chi_{[t_{k-1},t_k)}(t), \tag{6}$$

where $a = t_0 < t_1 < \cdots < t_n = b$ is a partition of $[a,b]$, and $g_k \in \mathcal{F}_{t_{k-1}} = \sigma(B_{t_{k-1}})$, in the spirit of taking the left endpoints as sample points. We define

$$I(g) = \int_a^b g(t,\omega)dB_t = \sum_k g_k(\omega)\left(B_{t_k} - B_{t_{k-1}}\right). \tag{7}$$

Exercise 7. *Show that* $E\left(I(g)^2\right) = E\left(\int_a^b |g|^2 dt\right)$.

Proof. We use the shorthand notation $\Delta B_k = B_{t_k} - B_{t_{k-1}}$ and $\Delta t_k = t_k - t_{k-1}$.

$$E\left(I(g)^2\right) = \sum_k E\left(g_k \Delta B_k\right)^2 + 2\sum_{k<j} E\left(g_k\Delta B_k \cdot g_j \Delta B_j\right)$$

Note that $g_k \in \mathcal{F}_{t_{k-1}}$, and ΔB_k is independent of $\mathcal{F}_{t_{k-1}}$, so the first term becomes

$$E\left(g_k\Delta B_k\right)^2 = Eg_k^2 \cdot E\left(\Delta B_k\right)^2 = \Delta t_k E f_k^2.$$

When $k < j$, then $k \leq j-1$, and thus $g_k, \Delta B_k, g_j \in \mathcal{F}_{t_{j-1}}$. The second term becomes

$$\begin{aligned} E\left(g_k\Delta B_k \cdot g_j\Delta B_j\right) &= E\left(E\left(g_k\Delta B_k g_j \cdot \Delta B_j | \mathcal{F}_{t_{j-1}}\right)\right) \\ &= E\left(g_k \Delta B_k g_j E\left(\Delta B_j | \mathcal{F}_{t_{j-1}}\right)\right) \\ &= 0. \end{aligned}$$

Therefore

$$E\left(I(g)^2\right) = \sum_k E\left(g_k^2 \Delta t_k\right) = E\left(\int_a^b |g|^2 dt\right).$$

□

It remains to note that every $f \in \mathcal{A}$ can indeed be approximated by simple functions:

Theorem 8 (See B. Øksendal for the proof)**.**

(1) Every $f \in \mathcal{A}$ *can be approximated in* L^2 *by simple functions* $\{g_n\}$ *of the type* (7)*. Thus one can define Itô integral*

$$\int_a^b f dB_t = \lim_{n\to\infty} \int_a^b g_n dB_t,$$

where the limit is taken in $L^2(\Omega, \mathbb{P})$*. Consequently,*

(2) ***Itô isometry.*** *For any* $f \in \mathcal{A}$*, we have* $E\left(\int_a^b f dB_t\right)^2 = E\left(\int_a^b |f|^2 dt\right)$