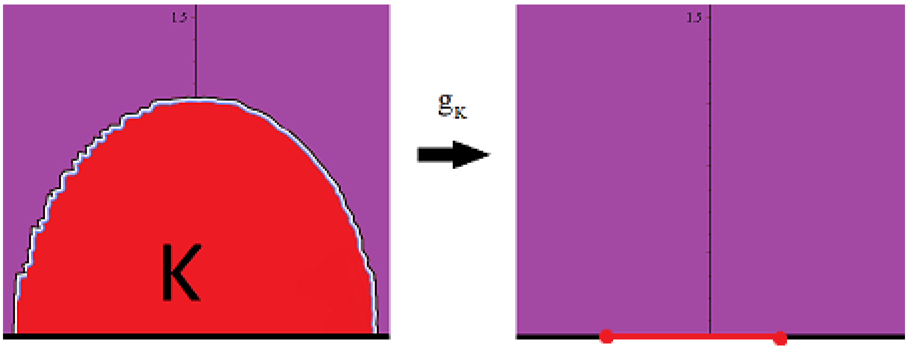


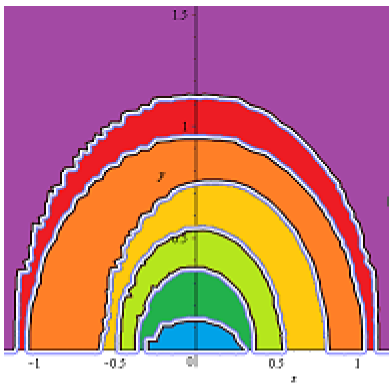
Recall that a *compact hull* K is a compact subset of $\bar{\mathbb{H}}$ with simply connected complement in the upper half-plane. We define g_k to be the unique conformal automorphism from $\mathbb{H} \setminus K \to \mathbb{H}$ satisfying the *hydrodynamic normalization*, namely that at infinity,

$$\text{Hydrodynamic Normalization:} \qquad g(z) = z + \frac{c_K}{z} + O(\frac{1}{z^2})$$

where c_K is the half-plane capacity which, by the residue theorem, is

$$\frac{1}{\pi}\int_{g(\partial K)} \text{Im}(g^{-1}(x))dx.$$

The integral expression for the half-plane capacity might lead one to wonder what sort of continuity properties we can exploit. For example, if we have a nested sequence of compact subsets ”behaving well” we might expect that their capacities reflect this good behavior.



It turns out that even more is true, but we need some preliminaries from complex analysis first, to which we now turn.

Let $\mathcal{S}$ be the collection of *schlicht* functions, i.e. conformal maps $f : \mathbb{D} \to \mathbb{C}$ normalized so that $f(0) = 0$ and $f'(0) = 1$. Because of the Riemann Mapping Theorem, the study of conformal automorphisms between simply connected domains (other than the entire plane) can be studied by studying $\mathcal{S}$. Perhaps the most important geometric result concerning these functions is the following:

Proposition 1 (Koebe Quarter Theorem) **:** *Let $f \in \mathcal{S}$. Then $B_{1/4}(0) \subset f(\mathbb{D})$.* ■

From the Quarter Theorem, the all-important *distortion theorem* follows:

Corollary 1.1 (Koebe Distortion Theorem) **:** *Let $f \in \mathcal{S}$ and let $|z| = r < 1$. Then*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}$$

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}$$

$$\frac{1-r}{1+r} \leq \left| z\frac{f'(z)}{f(z)} \right| \leq \frac{1+r}{1-r}$$

For a proof of these theorems see e.g. *Univalent Functions* by Peter Duren. ■

We can think of $\mathcal{S}$ as sitting in the metric space $\mathrm{Hol}(\mathbb{D})$ of all holomorphic functions on $\mathbb{D}$ with the topology of uniform convergence on compact subsets. Recall that a family $\mathcal{F}$ of holomorphic functions on a region Ω is normal if and only if it has compact closure in $\mathrm{Hol}(\Omega)$ if and only if it forms a locally bounded family. From the Distortion Theorem, we have the following results:

Corollary 1.2 **:** *$\mathcal{S}$ is a normal family which is closed in* $\mathrm{Hol}(\mathbb{D})$.

Proof: If B_r is the closed ball of radius $r < 1$ centered at zero, then by the Distortion Theorems, $|f(z)| \leq \frac{r}{(1-r)^2}$ on $B_r(0)$ for all $f \in S$.

Since these balls exhaust the unit disc, S is a locally bounded family and thus normal. Given a normal limit $f_n \to \tilde{f}$, uniformity gives $\tilde{f}(0) = 0$ and by Weierstrass, $\tilde{f}$ is analytic and $f_n' \to \tilde{f}'$ uniformly on compact subsets; thus $\tilde{f}'(0) = 1$. By Hurwitz's Theorem, $\tilde{f}$ is one-to-one, so $\tilde{f}$ is schlicht. Thus S is closed. ■

Corollary 1.3 **:** *Let $0 < r < 1$ and $f_n \to f \in \mathcal{S}$. Then $\partial(1/r f_n(r\mathbb{D})) \to \partial(f(\mathbb{D}))$.*

Proof: By the chain rule $1/r f(rz) \in S$ and since S is closed, $1/r f_n(rz) \to 1/r f(rz) \in S$ whenever $f_n \to f$. In particular, this must be the unique limit of

$$\lim_{\substack{n\to\infty \\ r\to 1}} \frac{1}{r} f_n(rz).$$

■

This convergence is only in the Hausdorff sense; for a complete treatment of the Hausdorff metric, see *Continuum Theory* by Sam Nadler. The interior of the closed topologist's sine curve as a codomain gives a counterexample to convergence in the strong curve topology.

Thus we have a case of the convergence of functions implying convergence of (the boundaries of) their codomains. Now we'd like to flip this around: What we would like to know is how much this geometric limiting of the image sets determines the behavior of a given family of holomorphic functions.

This would be particularly useful when studying SLE, since our object of interest is an increasing curve - or rather, the complement in $\mathbb{H}$ of this curve, and of course all of our theorems thus far can be transplanted from the disc to $\mathbb{H}$ by the Cayley Transform. It turns out that in the univalent case, much can be said in this regard.

Definition 2.1: Fix a $w_0 \in \mathbb{C}$. Suppose (D_n) is a sequence of simply connected regions satisfying $w_0 \in D_n$ for all $n \in \mathbb{N}$.

- (E1) If w_0 is not in the interior of $\cap D_n$, define the *Carathéodory Kernel with respect to* w_0 to be ker$(D_n) = \{w_0\}$.

- (E2) Otherwise, define ker$(D_n) = D$, where $D \neq \mathbb{C}$ is the largest connected open set U containing w_0 satisfying the following property $\mathcal{P}$:

  $\mathcal{P}$: for all compact $K \subset U$, there exists an $N_K \in \mathbb{N}$ so $K \subset D_n$ for all $n \geq N_K$.

That is to say, D is the largest domain containing w_0 whose compact subsets are eventually contained in all the D_n's; this is very similar to taking the infimum of a sequence of sets, but in this case it is "with respect to" a marked point. It is in this sense that D is a "limit" of the D_n's. We don't want to allow $\mathbb{C}$ for technical reasons, although in some treatments it is admissible. In case $D = \mathbb{C}$ we proclaim that the kernel is simply $\{w_0\}$.

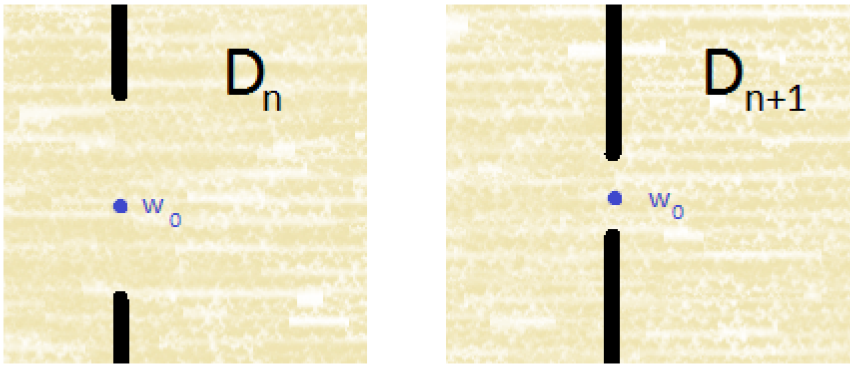
Note that D is well-defined, since if E_α is the collection of all open sets which are maximal with respect to $\mathcal{P}$ then their union would also be connected (since they all contain w_0) and open, so that $D = \cup E_\alpha$.

In general this definition is a bit convoluted, so let's look at some examples:

1. If D_n is an increasing sequence of domains containing w_0, then ker$(D_n) = \cup D_n$ if this union is not all of $\mathbb{C}$, and is w_0 otherwise. In the case of such an increasing sequence, D will be simply connected since any loop, a compact set, in D resides in only finitely many of the (simply connected) sets D_n, and since they are nested it resides in a single such set where it can be contracted.

2. If $D_n = \mathbb{D} + 1/n$ then ker$(D_n) = \mathbb{D}$ for $w_0 = 0$. Similarly, if we take the sequence of right half-planes bounded by the lines $x = \frac{1}{n}$, then this sequence converges to the right half-plane for any point contained in it. It converges to $\{0\}$ with respect to 0.

3. If $D_n = \mathbb{D} + (1 - 1/n)$ then ker$(D_n) = \{0\}$ with respect to 0, while ker$(D_n) = B_1(1)$ with respect to 1. Similarly, the sequence of right half-planes bounded by $x = -\frac{1}{n}$ converges to the same set as the previous example with respect to *any* point.

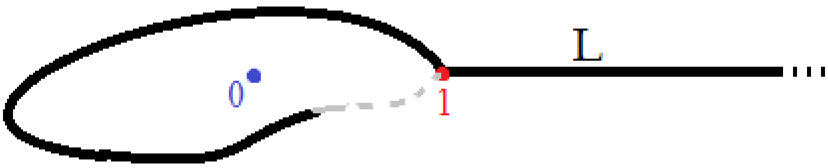


Example 4

4. If we let D_n be the region whose boundary is shown in black and let w_0 be the limit point of the boundary components, then $\ker(D_n) = w_0$ since w_0 is not contained in the interior of the intersection. If we take the kernel with respect to a different point, either on the left or right half-plane, then the kernel is the half-plane containing that point.

5. The following example will be the most important for us, namely that of an increasing simple curve which closes on itself. This will be relevant when we look at SLE$_\kappa$ for $\kappa > 4$ whose trace is given by a percolation process - in particular it will be self-intersecting, and by homogeneity of SLE it will do so instantaneously. Later on we will see that for $\kappa \leq 4$ the trace of SLE$_\kappa$ is almost surely simple, however.

   Let $L = [1, \infty)$ and let $\gamma : [0, 1] \to \mathbb{C}$ be a Jordan curve based at 1 so that zero lies in J, the bounded component of its complement. If $D_n = (L \cup \bar{\gamma}[0, 1 - {}^{1}/_{n}])^c$,



   then $\ker(\{D_n\}) = J$. In particular, $\ker(D_n)$ is **not** the intersection of the zero-components of the D_n's. Rather, here it is the interior of the zero-component of $\cap D_n$. This is in some sense typical, which we shall see shortly.

Definition 2.2 We say that a sequence of simply connected regions D_n containing w_0 *converges to its kernel*, written $D_n \to \ker(D_n)$, if every subsequence D_{n_k} has the same kernel. In all of the above examples, it can be verified that D_n converges to its kernel.

However, this is not always the case. Consider the following sequence of sets:

$$D_n = \begin{cases} B_1({}^{1}/_{2}) & \text{if } n \text{ is even} \\ B_1(-{}^{1}/_{2}) & \text{if } n \text{ is odd} \end{cases}$$

which alternates back and forth between the unit balls centered at $\pm 1/2$. Then with $w_0 = 0$ $\ker(D_{2n}) = B_1(1/2)$, $\ker(D_{2n+1}) = B_1(-1/2)$, so that D_n does not converge to its kernel (in fact, the kernel is the intersection of the two discs). Intuitively, the problem is that the sets "jump around"; materially, in the first five cases the boundaries of the regions converge in the Hausdorff sense, whereas here they do not.

Now we come to the principle result concerning convergence in the above sense. This theorem will allow us to develop the continuity properties necessary for the Loewner Equation, the main object of our study. We state it for $w_0 = 0$ and domain $\mathbb{D}$ as a matter of convenience, but any point and any simply connected domain (other than $\mathbb{C}$) would do.

Theorem 2.3 (Carathéodory Kernel Theorem): *Let D_n be a sequence of simply connected domains about 0 with $D_n \neq \mathbb{C}$ for all n. Let $f_n : \mathbb{D} \to D_n$ be the conformal isomorphism satisfying $f_n(0) = 0$ and $f_n'(0) > 0$ for all n. Let $\ker(D_n) = D \neq \{0\}, \mathbb{C}$. If $f : \mathbb{D} \to D$ is the conformal isomorphism also satisfying $f(0) = 0$ and $f'(0) > 0$, then D_n converges to its kernel if and only if $f_n \to f$ uniformly on compact subsets.*

Proof: First suppose that $f_n \to f$. If the kernel is $\{0\}$ then for no $r > 0$ is the closed disc of radius r about 0 contained D_n for all n, and thus $\mathrm{dist}(0, \partial D_n) \to 0$. Applying the distortion theorem to $\frac{f_n(z)}{f_n'(0)} \in \mathcal{S}$ gives

$$|f_n(z)| \leq \frac{|z|}{(1-|z|)^2} |f_n'(0)|$$

By the quarter theorem,

$$\frac{1}{4} |f_n'(0)| \leq \mathrm{dist}(0, \partial D_n)$$

Combining gives the result. Conversely if $f_n \to 0$ then $\ker(D_n)$ contains no disc about 0, for otherwise $f_n'(0) \geq 1/r$, contradicting normality. Then $\ker(D_n) = \{0\}$. Thus we have proved the theorem in the case $\ker(D_n) = \{0\}$.

Now assume $f_n \to f$ induces a non-trivial kernel D. If $K \subset f(\mathbb{D})$ is compact there is an r, $0 < r < 1$, with $K \subset f(B_r(0))$. $\Gamma = f(\partial B_r(0)) \implies \mathrm{dist}(K, \Gamma) = d > 0$. By normal convergence $\exists N_K$ such that

$$|f_n(re^{i\theta}) - f(re^{i\theta})| < \frac{d}{2}$$

for all $n > N_K$. Letting $\Gamma_n = f_n(\partial B_r(0))$ we have that Γ and Γ_n eventually have the same winding number about K, i.e. each value of K is obtained precisely once by each f_n by the argument principle. Thus $K \subset D_n$ for $n > N_K$ and so by definition of the kernel, $f(\mathbb{D}) \subset D$.

To show $D \subset f(\mathbb{D})$ let $0 \neq w \in D$ and take closed discs A, B with respective radii r, s about $w, 0$ in D. Let $C = A \cup B$. Then C is eventually contained in all D_n so that $\phi_n = f_n^{-1}$ are defined on C, and satisfy $|\phi_n(w)| < 1, \phi_n(0) = 0$, i.e. are a normal family on the interior of C. Taking a convergent subsequence ϕ_{n_k} we get ϕ satisfying the same normalization and thus $|\phi(w)| < 1$ by max mod, say $\phi(w) = z_0 \in \mathbb{D}$. Since $f_{n_k} \to f$ *uniformly* around z_0 we see that

$$\phi_{n_k}(w) \to \phi(w) \implies f_{n_k}(\phi_{n_k}(w)) \to f(\phi(w))$$

i.e. $w = f(z_0) \in f(\mathbb{D})$, establishing the reverse inclusion.

Thus combining we obtain that $f(\mathbb{D}) = D$, not all of $\mathbb{C}$ since its inverse would contradict Liouville's Theorem. Since every subsequence of f_n converges normally to f it follows that every subsequence of D_n has kernel D. But that's saying that $D_n \to \ker(D_n) = D$. Thus we have proven one direction of the theorem.

Now we assume D_n converges to its kernel D, with D not all of $\mathbb{C}$, and by previous commentary also not $\{0\}$. Picking small enough r we have that ϕ_n are eventually defined on $B_r(0)$. Since $f_n(0) \equiv 0$ we may define functions

$$\psi_n(z) = \phi_n(rz) \qquad \psi_n : \mathbb{D} \to \mathbb{D}$$

Note $\psi_n(0) = 0$ so by the Schwarz lemma,

$$r\phi_n'(0) = \psi_n'(0) \leq 1 \implies \psi_n'(0) \leq \frac{1}{r}.$$

Thus $f_n'(0) \geq r > 0$ are bounded below; if they were unbounded from above pick a subsequence f_k with derivatives at 0 going to ∞. Now,

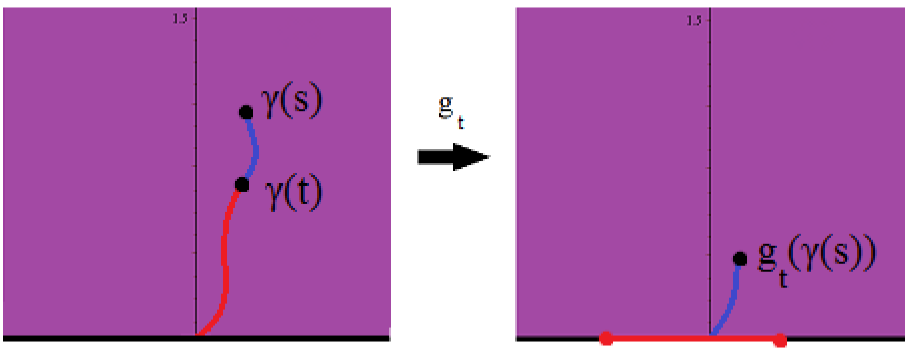
$$\text{dist}(0, \partial D_k) \geq \frac{1}{4} f_k'(0) = d_k \implies B_{d_k}(0) \in D_k, k \in \mathbb{N}$$

implying that $D = \mathbb{C}$, a contradiction. Therefore $|f_n'(0)|$ is bounded from above and below, and in particular is a normal family by local boundedness. If the family had another normal limit, call it g, then picking subsequences D_m, D_k we would find, from the first direction of the theorem, that $\ker(D_m) = f(\mathbb{D})$, $\ker(D_k) = g(\mathbb{D})$. Since by assumption we have that $D_n \to \ker(D_n)$ we have that $f(\mathbb{D}) = g(\mathbb{D})$. But both satisfy the Riemann Mapping Theorem normalization and thus $f = g$ by the uniqueness hypothesis of RMT. ∎

Corollary 2.4 (Radó Convergence Theorem): *Suppose in addition that D_n are Jordan regions bounded by Jordan curves J_n with $D_n \to \ker(D_n) = D$ and ∂D is a Jordan curve J. Then f_n converges uniformly on the* *closed* *disc if and only if for all $\epsilon > 0$ there exists N_ϵ with, for $n > N_\epsilon$, a continuous injection between J_n, J such that the distance between any two point on J_n and the corresponding point of J is less than ϵ.*

Proof: See Goluzin, p. 59-62. ∎

Now consider a simple curve $\gamma : [0, s] \to \mathbb{C}$ lying in the closed upper half-plane so that its intersection with $\mathbb{R}$ is solely $\gamma(0) = 0$, and let $0 \leq t < s$.



Here $g_t := g_{\gamma([0,t])}$ in our previous notation, i.e. g_t maps the complement of $\gamma([0,t])$ to $\mathbb{H}$. Define the complement of $\gamma([0,t])$ to be $\mathbb{H}_t$. Note that in this case, as $t \to s$ from either above or below we have that $\mathbb{H}_t \to \ker(H_s)$ for any sequence $t \le t_1 \le t_2 \le \cdots \to s$.

Then by the Carathéodory Kernel Theorem,

Corollary 2.6: *$g_t \to g_s$ uniformly on compact subsets on $\mathbb{H}_s$, as t approaches s.* ■

However, to get convergence at ∞ we have to work a bit harder:

(Exercise): Use the Schwarz Reflection Principle on $\bar{\mathbb{H}} \setminus g_s(\gamma([0,s]))$ to consider ∞ as being in the interior, then apply the Carathéodory Kernel Theorem.

As a corollary of the exercise, we finally prove what we set out for:

Corollary 2.7: *If $c_t := \text{hcap}(\gamma[0,t])$, then $c_t \to c_s$. Since half-plane capacity is increasing for increasing K_t, we have $c_t \nearrow c_s$.* ■

Before proceeding to the Loewner Equation we need one more result from geometric function theory, namely the *Beurling Projection Theorem*. This theorem will help us consolidate some of the complex-analytic and probabilistic aspects of the SLE theory. For the remainder of this section, let B_t denote Brownian motion in $\mathbb{C}$.

Lemma 3.1: *Let $\Omega \subset \mathbb{C}$ be a region such that Ω^c has a connected component containing more than one point in $\hat{\mathbb{C}}$. Let $\tau = \inf\{t \mid B_t \notin \Omega\}$, and let $f : \partial\Omega \to \mathbb{R}$ be measurable.*

1. *$P_z(\tau < \infty) = 1$ for $z \in \Omega$, P_z the escape-time distribution for B_t starting at z.*
2. *$u(z) = \mathbb{E}[f(B_t)]$ exists and is harmonic on $\partial\Omega$.*
3. If *f is continuous at $\zeta \in \partial\Omega$ and there is a connected set $C_\zeta \subset \Omega^c$ containing ζ and at least one other point, then $u(z) \to f(\zeta)$ as $z \to \zeta$.*

Proof Sketch: (1) With regard to SLE we are only interested in regions which are univalent images of $\mathbb{D}$. By rotational invariance of B_t, each arc of positive length on $\partial\mathbb{D}$ subtends a positive probability of being the escape point for B_t started at a $z \in \mathbb{D}$. By conformal invariance (the Laplacian is conformal in dimension 2) the result follows.

(2) Fix z and r such that $z + r\bar{\mathbb{D}} \subset \Omega$. Consider the stopping time $T = \inf\{t \mid |B_t - z| = r\}$. By the strong Markov property of B_t we get $\mathbb{E}[u(z) \mid B_{\cdot \wedge T}] = u(B_T)$. Taking expectation again yields

$$u(z) = \mathbb{E}[u(B_T)] = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta$$

by rotational invariance; in particular $u(z)$ satisfies the mean value property and is thus harmonic.

(3) Fix $\epsilon > 0$ and pick $\delta < \text{diam}(K_\zeta)$. Then $|f(\zeta) - f(w)| < \epsilon$ for $w \in \partial\Omega \cap B_\delta(\zeta)$. If $z \to \zeta$ then $P_z[B_t \text{exits before hitting} K_\zeta] \to 0$ by the first part. On the complement of this event, $B_\tau \in B_\delta(\zeta)$, so $|f(B_\tau) - f(\zeta)| < \epsilon$. Upon taking expectations, we arrive at the following inequality:

$$\mathbb{E}[\ |f(B_\tau) - f(\zeta)|\ ] \le \epsilon + 2||f||P_z[B_t \text{ exits before hitting } K_\zeta] \to 0$$

as $z \to \zeta$. Namely, $u(z) \to f(\zeta)$ and thus $u \in C(\bar{\Omega}), u \equiv f$ on $\partial\Omega$. ■

A few comments are in order. Parsing this lemma into words, the expected value of the location of Brownian motion is harmonic (compare to $\Omega = \mathbb{D}$ and $f(e^{it}) = sin(t), 0 \le t \le 2\pi$, in which case the statement is simply rotational invariance), and under some mild conditions on the boundary of Ω, if f has a continuous extension to a point on its boundary then the expectation converges to that value. As alluded to, the case for higher dimensions needn't hold since the Laplacian is no longer harmonic there, and in particular B_t fails to be conformally invariant. However, the most important observation is the following (immediate) corollary:

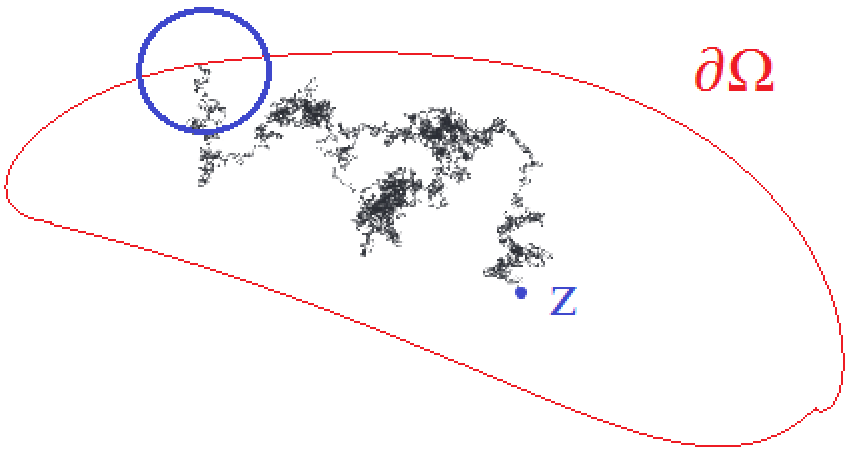
Corollary 3.2: *Assume for every ζ there is a C_ζ as described. Then $u(z)$ is the solution to the Dirichlet Problem for f on Ω.* ■

For applications to SLE, the existence of such sets will be trivial, thankfully.

Definition 3.3: Now we turn our attention to the concept of harmonic measure on the boundary of a connected region $\Omega \in \mathbb{C}$, one whose complement contains a non-trivial component as before. Namely, for a chosen $z \in \Omega$, define the *harmonic measure* μ_z on the Borel sigma-algebra $\mathcal{B}$ of $\partial\Omega$ to be

$$\mu_z(A) = P_z[B_\tau \in A], \qquad A \in \mathcal{B}$$

In other words, the measure of a subset of $\partial\Omega$ is equal to the probability that Brownian motion starting at z exits from Ω through the set A. For a trivial example, if $z = 0, \Omega = \mathbb{D}$ and A is an arc with length $2\pi\alpha$, we have $\mu_0(A) = \alpha$. In fact, it is intuitively clear that 0 is the only point for which this will be true for *any* non-trivial arc. The existence of harmonic measure follows from a straight-forward application of the Riesz Representation Theorem to Lemma 3.1. Refer to *Harmonic Measure* by Marshall and Garnett.



As is often the case, the trivial example is the most important one. For us, however, it's basically the only one: Recall that the (classical) Carathéodory Theorem gives us a continuous extension to the entire boundary for a conformal mapping of the disc onto a Jordan region. So by Lemma 3.1 (alt: by uniqueness of the solution to the Dirichlet Problem),

Corollary 3.4: *If Ω is a Jordan region and $f : \mathbb{D} \to \Omega$ is the Riemann map sending $0 \mapsto z_0$, then $\mu_{z,\Omega}(A) = \mu_{0,\mathbb{D}}(f^{-1}(A))$.* ■

Now for the denouement. The next two proofs are from Marshall and Garnett, *Harmonic Measure*:

Theorem 3.5 (Beurling Projection Theorem): *Let $E \subset \mathbb{D} \setminus \{0\}$, $E^* = \{|z| \mid z \in E\}$ be its radial trace. Write $V = \mathbb{D} \setminus E$ and $V^* = \mathbb{D} \setminus E^*$. Then for all $z \in \mathbb{D}$,*

$$\mu_{z,V}(E) \geq \mu_{-|z|,V^*}(E^*)$$

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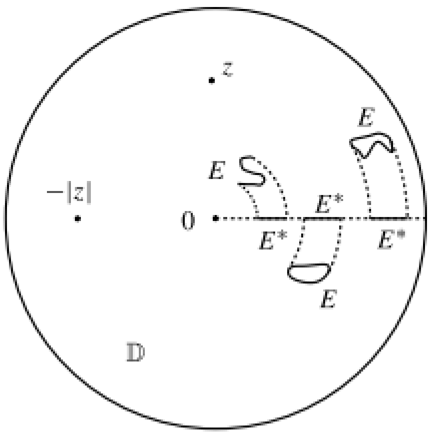


Circular Projection in $\mathbb{D}$.

Proof:

$$\mu_{z,V}(E) = \int_{\partial E} g(z,\xi)d\sigma(\xi)$$

by Green's Theorem, where g is the Green's Function for $\mathbb{D}$. Assume we have taken outward-looking normal vectors so that $\sigma \geq 0$. Consider the following measure and its Green potential:

$$\sigma^*(A) = \sigma\{z \in E : |z| \in A\}), \qquad G(z) = \int_{(\partial K)^*} g(z, |\xi|)d\sigma^*$$

From

$$g(-|z|, |\xi|) \leq g(z,\xi) \leq g(|Z|, |\xi|),$$

it follows that

$$G(-|z|) \leq \mu_{z,V}(K) \leq G(|z|).$$

Letting $z_n \to z$, by the second inequality $\liminf\{G(|z_n|)\} \geq 1$; by a comparison, $\mu_{z,V^*}(E^*) \leq G(z)$. Combining gives

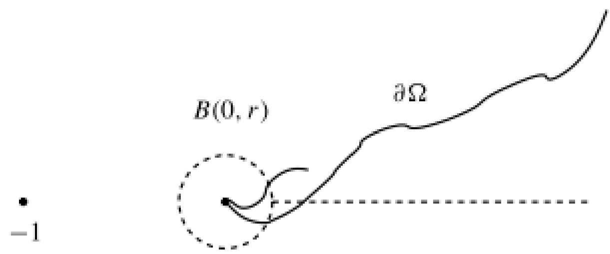
$$\mu_{-|z|,V^*}(E^*) \leq G(-|z|) \leq \mu_{z,V}(E)$$

which is the desired result. ∎

Corollary 3.6 (Beurling's Estimate): *Let Ω be a simply connected region, $z \in \Omega, \zeta \in \partial\Omega$. Then for r sufficiently small,*

$$\mu_{z,\Omega}(B_r(\zeta) \cap \partial\Omega) \leq C \cdot r^{\frac{1}{2}}.$$

Proof:



Without loss of generality assume $\xi = 0$ and $z = -1$. Project $\partial\Omega \setminus B_r(0)$ to $\mathbb{R}^+$. Set $X = \mathbb{C} \setminus (B_r(0) \cup [r, \infty)))$. Now apply Beurling's Theorem to $\hat{\mathbb{C}} \setminus B_r(0)$ to obtain

$$\mu_{z,\Omega}(B_r(0) \cap \partial\Omega) \leq \mu_{-1,X}(\partial B_r(0))$$

By a calculation,

$$\mu_{-1,X}(\partial B_r(0)) = \frac{2}{\pi} \arg\Big(\frac{i - \sqrt{r}}{i + \sqrt{r}}\Big) = \frac{4}{\pi} \tan^{-1}(\sqrt{r}) \propto r^{\frac{1}{2}} \blacksquare$$

Now we address the Loewner Equation. Reparametrize $\gamma(x)$ so that

$$g_t = z + \frac{2t}{z} + O(\frac{1}{z^2})$$

where the factor of two is coming from a difference between the slit and chordal SLE processes. Let χ denote the trace of SLE in the plane, and let γ be a simple curve as before.

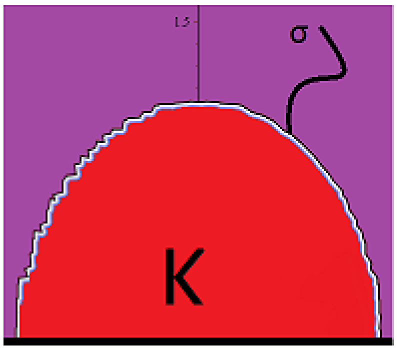
Theorem 4.1: *Let $\lambda(t) = g_t(\gamma(t))$. Then $\lambda(t)$ exists for all t, is continuous in t and the function $t \mapsto g_t(z_0)$ is differentiable for $z_0 \notin \chi$. Further,*

$$\textbf{Loewner's Equation:} \qquad \frac{d}{dt} g_t(z) = \frac{z}{g_t(z) - \lambda(t)}$$

This is a major theorem and will be broken into several lemmas. The simplicity of the equation, from which so much complexity arises, is startling, and we will see later that for SLE_κ, we have $\lambda(t) = \sqrt{\kappa} \cdot B_t$. The first step in the proof is the following:

Lemma 4.2: *Let K be a compact hull with* $\text{diam}(K) \leq 1$, *and let σ be a connected set intersecting K in at least one point. If* $\text{diam}(\sigma) \leq 1$ *then* $\text{diam}(g_K(\sigma)) \leq C \cdot (\text{diam}(\sigma))^{1/2}$ *regardless of K or σ.*

Proof: TBD.



The case we are most interested in is when both K and σ are parts of a simple curve. By the above lemma, as $t \to s$ the diameter of $\sigma = \gamma[t, s]$ converges to 0, so that:

Corollary 4.3:

$$\lambda(t) = \lim_{\substack{z \to \gamma(t) \\ z \in H_t}} (g_t(z)) \text{ exists.} \quad \blacksquare$$

Lemma 4.4 (Loewner's Lemma): *If $x \in \mathbb{R} \setminus K$ then* $0 < g'_K < 1$ *regardless of K.*

Proof: TBD.

In particular the derivative can't change direction, so we obtain that $g_K(x) < x$ on the left of K and $g_K(x) > x$ on the right. Since $\lambda(t)$ is contained in $g_s(\sigma)$ for t approaching s by the above lemma, then by Lemma 4.2 $\lambda(t)$ is continuous in t.