Introduction to SLE Lecture Notes

May 13, 2016 -

The goal of this section is to find a sufficient condition of λ for the hulls K_t to be generated by a simple cure. It turns out if $||\lambda||_{\frac{1}{2}} < 4$ then K_t is generated by a simple curve. We will prove this fact in this section.

Proposition. *If there exist a continuous function $v : \mathbb{R}^+ \to \mathbb{R}^+$ such that $v(0) = 0$ and for all t we have*

$$\int_0^\epsilon |(g_t^{-1})'(\lambda_t + iy)|dy \leq v(\epsilon),$$

then

$$\beta(t) := \lim_{\epsilon \to 0} g_t^{-1}(\lambda_t + i\epsilon)$$

exists and is continuous, and the hulls are generated by a curve, namely $\beta(t)$.

Proof. $\lim_{y\to 0} g_t^{-1}(\lambda_t + iy)$ exists because the length of $\gamma := g_t^{-1}(i[0, y])$ is finite. Precisely, we can prove this by contradiction; if $\lim_{y\to 0} g_t^{-1}(\lambda_t + iy)$ does not exist, then γ has at least 2 distinct limit points. γ travels back and forth between these two points, so it has infinite length. Moreover,

$$|\beta(t) - g_t^{-1}(\lambda_t + i\epsilon)| \leq v(\epsilon), \forall t.$$

Since the map $t \mapsto g_t^{-1}(\lambda_t + i\epsilon)$ is continuous (Exercise), $\exists \delta_\epsilon > 0$ s.t.

$$|g_t^{-1}(\lambda_t + i\epsilon) - g_t^{-1}(\lambda_s + i\epsilon)| \leq v(\epsilon)$$

if $|t-s| \leq \delta_\epsilon$. Then

$$|\beta(t)-\beta(s)| \leq |\beta(t)-g_t^{-1}(\lambda_t+i\epsilon)|+|g_t^{-1}(\lambda_t+i\epsilon)-g_s^{-1}(\lambda_s+i\epsilon)|+|\beta(s)-g_s^{-1}(\lambda_s+i\epsilon)| \leq 3v(\epsilon).$$

If $v(\epsilon) \to 0$ as $\epsilon \to 0$, then β is continuous.

□

Remark 1. *A remark here is that, if*

$$|g_t^{-1\prime}(\lambda + iy)| \leq Cy^{-\alpha}$$

for some $\alpha < 1$, all t and $y < 1$, then the assumptions in the proposition above is fulfilled because

$$\int_0^\epsilon |(g_t^{-1})'(\lambda_t + iy)|dy \leq C \int_0^\epsilon y^{-\alpha} dy = C'\epsilon^{1-\alpha}.$$

Then β is continuous and generates the hull.

Theorem (By Marshall-R, Lind, R-Tran, Zinsmeister)**.** *If $||\lambda||_{\frac{1}{2}} < 4$ for all t and s, then the L.E. generates simple curves. Moreover, the curve is a quasiconformal arc.*

Note.

1. $||\lambda||_{\frac{1}{2}} < 4 \Leftrightarrow |\lambda(t) - \lambda(s)| < 4|t-s|^{\frac{1}{2}}$ *for all t and s.*

2. *If $\phi : \mathbb{D} \to \mathbb{D}$ analytic, then $|\phi'(z)| \leq \frac{2}{1-|z|}$.*

3. *g_t satisfies the ODE $\dot{g}_t = \frac{2}{g_t - \lambda_t}$ but g_t^{-1} does not satisfy any ODE. Instead we can write it in terms of a PDE, namely $g(t, g_z^{-1}(z)) = z$.*

4. *$t \mapsto B_{T-t} - B_T$ is a Brownian motion.*

Proof. (Sketch) The idea here is to show the Holder continuity: $\exists \alpha = \alpha(||\lambda||) < 1$ s.t.

$$|(g_t^{-1})'(\lambda_t + iy)| \leq \frac{C}{y^\alpha}.$$

Then by remark 1 and proposition above we have the desire conclusion. First we try to set up the formula for the backward flow of g_t. Fix time $T > 0$, let $\tilde{\lambda}_t = \lambda_{T-t}$ (The tilde might be dropped later). Consider the backward flow, $g_{T-t}(z)$. By the chain rule,

$$\partial_t g_{T-t}(z) = \frac{-2}{g_{T-t}(z) - \lambda_{T-t}}.$$

Thus if satisfies the ODE $\dot{f}_t(z) = \frac{-2}{f_t(z) - \tilde{\lambda}}$ with the initial condition $f_0(z) = z$, then $f_T = g_T^{-1}$. Note that it is not true in general that $f_t = g_t^{-1}$.

Now we apply a translation so that $\tilde{\lambda}_t$ stays fixed, namely, $Z_t = X_t + iY_t = f_t(z) - \tilde{\lambda_T}$. Then we have

$$\partial_t f_t(z) = \partial_t(Z_t + \tilde{\lambda}_t) = \frac{-2}{Z_t} = \frac{-2(X_t - iY_t)}{X_t^2 + Y_t^2}.$$

Taking the real and imaginary part we have

$$\partial_t(X_t + \tilde{\lambda}_t) = \frac{-2X_t}{X_t^2 + Y_t^2} \tag{1}$$

$$\partial_t Y_t = \frac{2Y_t}{X_t^2 + Y_t^2} > 0, \tag{2}$$

since $Y_t > 0$. Note that (2) implies we are flowing upward as we hoped.

Now consider $\partial_t \log|f_t'(z)|$.

$$\begin{aligned}\partial_t \log|f_t'(z)| &= \partial_t \operatorname{Re}\log f_t'(z) = \operatorname{Re}\partial_t \log f_t'(z) = \operatorname{Re}\frac{\partial_t f_t'(z)}{f_t'(z)} \\ &= \operatorname{Re}\frac{\partial_z \partial_t f_t(z)}{f_t'(z)} = \operatorname{Re}\frac{\partial_z \frac{-2}{f_t(z)-\tilde{\lambda}_t}}{f_t'(z)} = \operatorname{Re}\frac{2}{Z_t^2}(\text{Quotient rule}) \\ &= 2\operatorname{Re}\frac{(X_t - iY_t)^2}{(X_t^2+Y_t^2)^2} = 2\frac{X_t^2 - Y_t^2}{(X^2+Y^2)^2}\end{aligned}$$

It follows that,

$$\begin{aligned}|f_t'(z)| &= \exp\log|f_t'(z)| \\ &= \exp\int_0^t \partial_s \log|f_s'(z)|ds \\ &= \exp\int_0^t \frac{X_s^2 - Y_s^2}{X_s^2+Y_s^2}(\frac{2ds}{X_s^2+Y_s^2})\end{aligned}$$

Let $u_s = \log Y_s$, then $\partial_s u_s = \frac{\partial_s Y_s}{Y_s} = \frac{2ds}{X_s^2+Y_s^2}$ by (2). Applying the substitution, we have

$$|f_t'(z)| = \exp\int_{u_0}^{u_t} \frac{(\frac{X_u}{Y_u})^2 - 1}{(\frac{X_u}{Y_u})^2+1}du = \exp\int_{u_0}^{u_t}\frac{W_u^2-1}{W_u^2+1}du,$$

where $W_u := \frac{X_u}{Y_u}$. If we assume $W_t < k$ for some k independent on t, then $\frac{W_u^2-1}{W_u^2+1} < \alpha$ for some $\alpha < 1$ (Consider the graph of function $\frac{x-1}{x+1}$). Then the theorem follows from Remark 1 since

$$|f_t'(z)| = \exp\int_{u_0}^{u_t}\frac{W_u^2-1}{W_u^2+1}du < \exp\int_{u_0}^{u_t}\alpha du = \exp(\alpha(u_t - u_s)) = e^{\alpha\log(\frac{Y_t}{Y_0})} = (\frac{Y_t}{Y_0})^\alpha.$$

□

Now it remains to show our assumption that $\exists k$ independent on t with $W_t < k$ for all t is true. To achieve this we introduce 2 lemmas as the following.

Lemma 1. *If* $||\tilde{\lambda}||_{\frac{1}{2}} \leq 4$ *and* $X_0 = 0$, *then* $\exists C = C(||\tilde{\lambda}||_{\frac{1}{2}})$, *s.t.* $|X_s| \leq CY_s$.

Lemma 2. *For every* $||\tilde{\lambda}||_{\frac{1}{2}} < 4$ *there exists* $c_1 < c_2$ *s.t. if* $\frac{x_0}{y_0} = c$ *and* $\frac{x_s}{y_s} \geq c_1, \forall s \in [0, T]$*, then* $\frac{x_t}{y_t} \leq c_2$ *for all t.*

Note.

1. *Lemma 1 implies our assumption above with* $\alpha = \frac{C^2+1}{C^2-1}$.

2. *Lemma 2 implies lemma 1. To see this, we use the "reseting clock argument". Take the last time, namely* t_0*, when* Z_t *escapes the cone* $\{y > c_1 x\}$*. Then* Z_{t-t_0} *satisfies the condition for lemma 2. By lemma 2, the conclusion of lemma 1 is true with* $C = c_2$*. So it suffices to show lemma 2.*

3. *Both lemmas hold as long as* $||\tilde{\lambda}||_{\frac{1}{2}} < 4$*. In class we gave a proof of lemma 2 for* $||\tilde{\lambda}||_{\frac{1}{2}} = \sqrt{2} - 1$*. Here we will give a direct proof of lemma 1 for* $||\tilde{\lambda}||_{\frac{1}{2}} < 2$*. The proof becomes rather technical for* $2 \leq ||\tilde{\lambda}||_{\frac{1}{2}} < 4$*. Readers who are interested may go through theorem 3.1 - 3.6 in this paper for more details.*

Proof. (of lemma 1)

Let $\sigma = ||\tilde{\lambda}||_{\frac{1}{2}} < 4$. First we claim that if $|X_t| < \sigma\sqrt{t}$ for some $\sigma < 2$ and all $t \geq CY_0^2$, then

$$Y_t \geq Lt$$

for all t, where $L = \min\left(\frac{1}{C}, 4 - \sigma^2\right) > 0$.

First assuming the claim. By equality (1), if $X_s >= 0$ then $X_t + \tilde{\lambda}_t$ decreases. By symmetry and considering the last time that X_s is of the same sign as X_t, we have

$$|X_t| \leq |X_0| + sup\{|\tilde{\lambda}_s - \tilde{\lambda}_t| : s \in [0, t]\} \leq 0 + \sigma\sqrt{t}.$$

Therefore the assumption of the claim is satisfied with arbitrarily small C. It flows from the claim that

$$\frac{|X_t|}{Y_t} < \frac{\sigma}{\sqrt{4-\sigma^2}},$$

for t greater than arbitrarily small C. So it is true for all t, which concludes the proof.

Next we show the claim by contradiction. Let $t_0 = CY_0^2$. Since $L \leq 1/C$ and Y_t is increasing, we have $Lt \leq t/C \leq Y_0^2 \leq Y_t^2$. If the claim were not true, then since $Y_0 > 0 = L\sqrt{0}$, there exists a minimal $s > t_0$ s.t. $Y_s^2 = Ls$ and $Y_t^2 \geq Lt$ on $[0, s]$. Then it follows from (2) that

$$\partial_t Y_t^2 = \frac{4Y_t^2}{X_t^2 + Y_t^2} \geq \frac{Lt}{M^2 t + Lt} \geq L$$

for all $t \in (t_0, s)$, which implies $Y_s^2 - Y_{t_0}^2 \geq L(s - t_0)$. This contradicts to our assumption that $Y_s^2 - Y_{t_0}^2 = Ls - Y_{t_0}^2 < L(s - t_0)$.

□