MATH 583 — Introduction to SLE

May 9 and May 11, 2016

For the last exercise $\int_0^t \mathcal{B}_s \, d\mathcal{B}_s$, let $f(x) = x^2$, then $Y_t = f(\mathcal{B}_t) = \mathcal{B}_t^2$, by Ito's formula,

$$dY_t = 2\mathcal{B}_t \, d\mathcal{B}_t + \frac{1}{2} \cdot 2 \, dt$$

then

$$\mathcal{B}_t^2 - t = \int_0^t dY_s - \int_0^t ds = 2 \int_0^t \mathcal{B}_s \, d\mathcal{B}_s.$$

Now, let's go back to complex. Recall that:

- The killed set (hulls) $(K_t)_{t \geq 0}$, are rather implicitly defined by $\dot{z}_t = \frac{2}{z_t - \lambda_t}$;
- when say $(K_t)_{t \geq 0}$ is generated by the curve $(\gamma)_{t \geq 0}$ if $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$;
- $\exists K_t$ s.t K_t is **not** generated by a curve. (think about the spiral example mensioned in earlier lectures)

The following is a very deep theorem which says that for SLE_κ, only good things can happen.

Theorem 1. *(R-Schramm $\kappa \neq 8$, Lawler-Schramm-Werner $\kappa = 8$) $\forall$ κ, SLE_κ is a.s. generated by (continuous) curves, ("SLE-trace").*

- *For $0 \leq \kappa \leq 4$, γ is a simple curve (a.s.);*

- *for* $4 < \kappa < 8$, $\bigcup_{t>0} K_t = \mathbb{H}$ *but* $\mathbb{P}[z \in \gamma[0,\infty)] = 0$;
- *for* $\kappa \geq 8$, $\gamma[0,\infty) = \mathbb{H}$.

Moreover, $dim_{\mathcal{H}}\gamma[0,\infty] = \begin{cases} 1+\frac{\kappa}{8}, & 0 \leq \kappa \leq 8 \\ 2, & \kappa > 8. \end{cases}$

Now focus on the continuity, how can we prove the continuity of the trace? First recall one classical fact:

Theorem 2. *(Carathéodory's theorem) Let* $f : \mathbb{D} \to G$ *(a simply connected domain) be the inverse of the Riemann map, then* f *extends continuously to* $\overline{\mathbb{D}} \iff \partial G$ *is locally connected.*

Theorem 3. *If* f *is analytic in* $\mathbb{D}$*, then* $\exists\ C, \tilde{C} > 0$ *and* $0 < \alpha < 1$ *s.t*

$$|f(z) - f(w)| \leq C|z-w|^{1-\alpha} \quad \forall z, w$$

if and only if

$$|f^{'}(z)| \leq \tilde{C}(1-|z|)^{-\alpha} \quad \forall z.$$

The proof is left to the reader as an exercise with the hint: for "if" direction, use Cauchy integral formula and for "only if" direction, take an integral along a clever choice of curve connecting $z,\ w$.

Theorem 4. *Let* $\lambda : [0,\infty) \to \mathbb{R}$ *be a continuous function,* g_t *be the hydrodynamic normalization from* $\mathbb{H}_t = \mathbb{H} \setminus K_t$ *to* $\mathbb{H}$*. If* $\beta(t) := \lim_{\epsilon \to 0} g_t^{-1}(\lambda_t + i\epsilon)$ *exists and is continuous, then* K_t *is generated by the curve* $\beta(t)$*.*

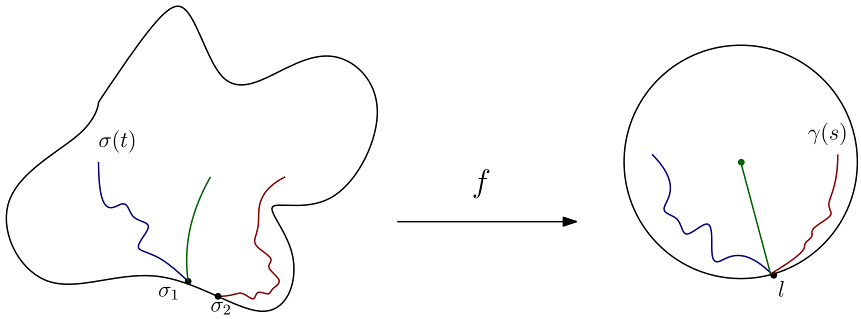
Lemma 5. *If* f *is a conformal map from a simply connected domain* H *to* $\mathbb{D}$*,* $\sigma : [0,1) \to H$ *and* $\sigma(t) \to \sigma_1 \in \partial H$ *as* $t \to 1$ *then*

(1) $f(\sigma(t)) \to l$ *as* $t \to 1$;

(2) $f^{-1}(rl) \to \sigma_1$ *as* $r \to 1$*;*

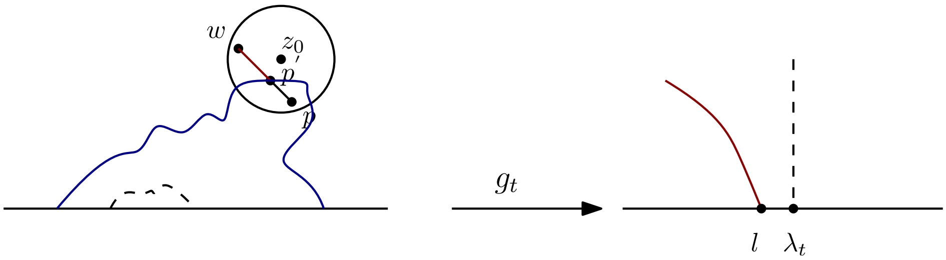
(3) if a curve $\gamma(s) \subset \mathbb{D}$ *ends at* l *and if* $f^{-1}(\gamma(s))$ *converges to* σ_2 *as* $s \to 1$*, then* $\sigma_2 = \sigma_1$*.*

Here is a sketch of the proof and the details are left to the readers who are interested.



Proof. For (1), fix $z_0 \in H$, consider the harmonic measure $\omega_H(z_0, \sigma(t,1))$, which is controlled by $(diam\sigma(t,1))^{\frac{1}{2}}$ by Beurling estimate. Since harmonic measure is conformal invariant, $\omega_H(z_0, \sigma(t,1)) = \omega_{\mathbb{D}}(f(z_0), f(\sigma(t,1)))$. Also, we know $\omega_{\mathbb{D}}(f(z_0), f(\sigma(t,1))) \approx diam f(\sigma(t,1))$, so it implies the diameter of $f(\sigma(t,1))$ goes to 0 as t goes to 1. Thus $f(\sigma(t))$ converges to some l. For (2), first show that $\omega(f(\sigma), rl, \mathbb{D}) \to \frac{1}{2}$ as $r \to 1$. That implies $\omega(f(\sigma), rl, \mathbb{D})$ is bounded away from 0 for r close enough to 1. If there exists a subconvergent sequence $\{f^{-1}(r_j l)\}$ which doesn't go to σ_1, then the limit point has positive distance from blue curve $\sigma(t)$, thus the harmonic measure $\omega(\sigma, f^{-1}(r_j l), H)$ goes to 0 which results in a contradiction. For (3), replace σ by $f^{-1}(\gamma)$ then apply part (1) to get that $f^{-1}(\gamma)$ and $f^{-1}(rl)$ will land at the same point, i.e $\sigma_1 = \sigma_2$. □

Now, let's use this lemma to prove Theorem 4.



Proof. Claim: Let $S(t)$ be the set of accumulation points of $g_t^{-1}(\lambda_t + z)$ as $z \to 0$. If $z_0 \in S(t_0)$, then $z_0 \in \beta[0, t_0]$.

pf of claim: Fix $z_0 \in S(t_0)$, then $z_0 \in \partial \mathbb{H}_{t_0}$ since conformal maps map boundary to boundary. Now fix $\epsilon > 0$, let t be the first time such that $K_t \cap \overline{D_\epsilon(z_0)} \neq \emptyset$, where $D_\epsilon(z_0)$ is the disk centered at z_0 with radius ϵ. We know that $t \leq t_0$ since $z_0 \in \partial \mathbb{H}_{t_0}$. Fix $w \in \mathbb{H}_{t_0} \cap D_\epsilon(z_0)$ and fix $p \in K_t \cap \overline{D_\epsilon(z_0)}$, then $\exists p^{'} \in K_t$ such that the line segment $\sigma = [\omega, p^{'}) \in \mathbb{H}_t$. Then apply Lemma 5 part (1), $g_t(\sigma)$ converges to some $l \in \mathbb{R}$ as approaching $p^{'}$. Then $l = \lambda_t$, otherwise, we can take a point q very close to $p^{'}$ which is mapped by g_t to a point away from λ_t, which implies that the Loewner equation will run longer and q won't be killed by time t. Then by part (2) of Lemma 5, $p^{'} = \beta(t) \in \beta[0, t_0]$. Let $\epsilon \to 0$, we obtain $z_0 \in \beta[0, t_0]$.

Now we have $S(t) = \bigcap_{\epsilon > 0} \overline{g_t^{-1}(D_\epsilon(\lambda_t))} \subset \beta[0, t]$. To show that (K_t) is generated by β, we need to show that $\mathbb{H}_t$ is the unbounded component of $\mathbb{H} \setminus \beta[0, t]$. (Remark: if $\beta(0, \infty)$ is a simple curve in $\mathbb{H}$ then this is equivalent to $K_t = \beta[0, t]$.) Let $A = \bigcup_{s \leq t} S(s)$, it's easy to see that $A = \beta[0, t]$. Notice that $A \subset K_t$ and $\partial \mathbb{H}_t \cap \mathbb{H} \subset A$ (left as an exercise). This implies that $A \cap \mathbb{H}_t = \emptyset$. Since $\mathbb{H}_t$ is connected, $\mathbb{H}_t$ is contained in one of the component of A^c, and it is actually one of the components since $\partial \mathbb{H}_t \subset A$. $\square$