MATH 583 — Introduction to SLE

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Last time, we defined the Ito integral $X_t = \int_0^t f(s,\omega)\ d\mathcal{B}_s$. By a theorem, there exists a process $t \mapsto \hat{X}_t$ continuous and $\forall t,\ \mathbb{P}(\hat{X}_t \neq X_t) = 0$. We can think of this as a process which only uses the information up to time t,

$$X_t = \mathbb{E}(\int_0^b f(s,\omega)\ d\mathcal{B}_s | \mathcal{F}_t).$$

This is a martingale.

Lemma. *Let $dX_t = a(t,\omega)dt + b(t,\omega)d\mathcal{B}_t$.*

$$(\iff X_t = X_0 + \int_0^t a(s,\omega)\ ds + \int_0^t b(s,\omega)\ d\mathcal{B}_s)$$

Then X_t is martingale if and only if $a(t,\omega) = 0$ for a.e. (t,ω) in the product space.

Proof. If $a(t,\omega) = 0$ a.e, then by the definition of Ito's integral, we always take the left point as sample points, then each $\Delta\mathcal{B}_k$ is independent of $\mathcal{F}_s$, so $\mathbb{E}[\int_s^t b(r,\omega)\ d\mathcal{B}_r | \mathcal{F}_s] = 0$. This implies $\mathbb{E}[X_t - X_s | \mathcal{F}_s] = 0$ which indicates X_t is a martingale.

On the other hand, if X_t is martingale, then $X_s = \mathbb{E}[X_t | \mathcal{F}_s]$ for any $s \leq t$.

That implies

$$\begin{aligned}0 &= \mathbb{E}[X_t - X_s|\mathcal{F}_s] \\ &= \mathbb{E}[\int_s^t a(r,\omega)\ dr + \int_s^t b(r,\omega)\ d\mathcal{B}_r|\mathcal{F}_s] \\ &= \mathbb{E}[\int_s^t a(r,\omega)\ dr|\mathcal{F}_s]\end{aligned}$$

Then by Lebesgue differentiation theorem we have $\mathbb{E}[a(t,\omega)\ dr|\mathcal{F}_s]$=0 $\forall s \leq t$, and then by Martingale convergence theorem, we'll get $a = 0$ a.e. □

Ito's formula. Let $f(t,x) \in C^2$, $dX_t = adt + bd\mathcal{B}_t$. Then $Y_t = f(t, X_t)$ is an Ito process and

$$dY_t = \partial_t f(t,X_t)dt + \partial_x f(t,X_t)dX_t + \frac{1}{2}\partial_{xx} f(t,X_t)(dX_t)^2,$$

where $(dX_t)^2 = b^2 dB_t$.

Proof. Applying Taylor's expansion, we have

$$\begin{aligned}f(t,X_t) - f(0,X_t) &= \sum_k f(t_k, X_{t_k}) - f(t_{k-1}, X_{t_{k-1}}) \\ &= \sum_k \partial_t f(t_{k-1}, X_{t_{k-1}})\Delta t_k + \partial_x f(t_{k-1}, X_{t_{k-1}})\Delta X_k \\ &\quad + \frac{1}{2}\partial_{xx} f(t_{k-1}, X_{t_{k-1}})(\Delta X_k)^2 + o(\Delta t_k + (\Delta X_k)^2).\end{aligned}$$

The first part gives $\int_0^t \partial_t f dt$. Notice $\Delta X_k = a_k \Delta t_k + b_k \Delta B_k$, so the second part gives $\int \partial_x f\ a\ dt + \int \partial_x f\ b\ d\mathcal{B}_t$. Now, consider the third part,

$$\sum \partial_{xx} f\ (a_k\Delta t_k + b_k \Delta B_k)^2 = \sum \partial_{xx} f\ (a_k^2(\Delta t_k)^2 + 2a_k b_k \Delta t_k \Delta B_k + b_k^2(\Delta B_k)^2)$$

Since $f \in C^2$, and a is continuous, so $\partial_{xx} f\ a^2$ is integrable, hence $\sum \partial_{xx} f\ (a_k^2(\Delta t_k)^2) \leq C \max_k |\Delta t_k|$ for some constant C; thus $\sum \partial_{xx} f\ (a_k^2(\Delta t_k)^2) \to 0$. To show $\sum \partial_{xx} f\ 2a_k b_k \Delta t_k \Delta B_k$ is small, we need to show its expectation is 0 and its variance is small. The expectation is 0 since ΔB_k and $\partial_{xx} a_k b_k$ are independent and $\mathbb{E}(\Delta B_k) = 0$. The variance is small since the mixed terms $i \neq j$ has expectation 0, and $i = j$ terms has $(\Delta t_j)^2$ which has very small expectation. The last thing we need to show is that $\sum \partial_{xx} f \; b_k^2((\Delta B_k)^2 - \Delta t_k)$ has expectation 0 and very small variance. Its expectation is 0 since $\mathbb{E}[(\Delta B_k)^2]$ is the variance of $\mathcal{B}_{t_k} - \mathcal{B}_{t_{k-1}}$ which is Δt_k and the independence of ΔB_k and $b_k^2 \partial_{xx} f$. The variance is very small, because the terms with $i < j$, $((\Delta B_j)^2 - \Delta t_j)$ is independent from other terms hence its expectation is 0; and for the terms with $i = j$, noticing that $(\Delta B_j)^4 \overset{d}{=} \mathcal{B}_{\Delta t_j}^4 \overset{d}{=} (\sqrt{\Delta t_j} \mathcal{B}_1)^4 = 3(\Delta t_j)^2$ and $(\Delta B_j)^2 \Delta t_j \overset{d}{=} (\Delta t_j)^2$, then

$$(\Delta B_j)^4 - 2\Delta (B_j)^2 \Delta t_j + (\Delta t_j)^2 \overset{d}{=} C(\Delta t_j)^2$$

for some constant C, which has very small expectation. $\square$

Exercise. $\int_0^t \mathcal{B}_s \; d\mathcal{B}_s \overset{d}{=} \frac{1}{2}\mathcal{B}_t^2 - \frac{1}{2}t$.