(Lecture 7, January 22, 2014 of Geometric Function Theory, Professor Steffen Rohde)

Recall that a homeomorphism f is quasisymmetric if

$$H(f,z,r) := \frac{max_{|z-w|=r}|f(z)-f(w)|}{min_{|z-w|=r}|f(z)-f(w)|} \leq H$$

for all z and for all r.

As a "challenge" homework: Find a QS map that is not absolutely continuous. We build up to the following

Theorem 1. *If f is QS on $\mathbb{R}^2$, then $f \in W^{1,2}_{loc}$, and $|\partial f| \leq k(H)|\overline{\partial} f|$.*

We begin with several definitions.

Definition 2. *We say that g is an upper gradient of f if for all $z, z' \in \mathbb{R}^2$ and for all γ joining z and z', we have*

$$|f(z)-f(z')| \leq \int_\gamma g(s)ds := \int_0^L g \circ \gamma(s) ds$$

where γ is rectifiable and the last integral is parametrization by arc length.

(Here Steffen says it's good to read the first chapters of "quasiconformal maps in $\mathbb{R}^n$")

Definition 3. *We let*

$$L^f_\epsilon(z) := sup_{|w-z|<\epsilon} \frac{|f(z)-f(w)|}{|z-w|}$$

which we take to be in $[0,\infty]$.

We can make an analogous definition with a domain G replacing $\mathbb{R}^2$.

Example 1. If f is smooth then $|\bigtriangledown f| \leq L_\epsilon \leq C|\bigtriangledown f|$ where C depends on the second derivatives.

Lemma 4. *L^f_ϵ is an upper gradient.*

Proof. Notice that L_ϵ is decreasing with ϵ, so that we may assume $\text{dist}(\gamma, \partial G) > \epsilon$. (Otherwise shrink ϵ). If $\text{diam}(\gamma) < \epsilon$, then for all $x \in \gamma$ we have

$$|f(z)-f(z')| \leq \frac{|f(z)-f(x)|}{|z-x|}|z-x| + \frac{|f(z')-f(x)|}{|z'-x|}|z'-x|$$
$$\leq L_\epsilon(x)(|z-x|+|z'-x|) \leq L_\epsilon(x) l(\gamma)$$

where $l(\gamma)$ is the length of the curve.

Now integrating both sides with respect to x, we have

$$l(\gamma)|f(z)-f(z')| \leq \int_\gamma L_\epsilon(x) l(\gamma)|dx|$$

From here we cancel $l(\gamma)$ from both sides. □

Lemma 5. *(Crucial lemma) For $\epsilon < r$ and $f : \Omega \to \Omega'$ H-quasisymmetric, we have*

$$|E_t| \leq \frac{C(H)}{t^2}|f(B(z_0, 2r))|$$

(where $B(z_0,2r) \subset \Omega$ and $E_t := \left\{\zeta : L^f_\epsilon(\zeta) > t, \zeta \in B(z_0,r)\right\}$, and $|\cdot|$ is Lebesgue measure.)

There are two ideas used in this proof:

1. $|f(z)-f(w)|^2 \asymp |f(B(z,|w-z|))|$, and
2. The following covering lemma: If $\mathcal{F}$ is a family of balls with uniformly bounded radii, then there is a disjoint subcollection $\{B_j\}$ such that $\bigcup_{B\in\mathcal{F}} B \subset \bigcup_j 5B_j$.

Proof. (crucial lemma): For all $z \in E_t$, there is a w_z such that $|w_z - z| < \epsilon$ and where $|f(z)-f(w_z)| > t|z-w_z|$. The covering lemma implies that there are z_j, w_{z_j} such that $B_j = B(z_j, |w_{z_j} - z_j|)$ are disjoint, but that $\bigcup_j 5B_j \supset E_t$. Thus

$$|E_t| \le \sum_j |5B_j| = 25\pi \sum_j |z_j - w_{z_j}|^2 \le$$

$$\frac{25\pi C}{t^2}\sum |f(z_j)-f(w_{z_j})|^2 \le \frac{C}{t^2}\sum |f(B_j)| \le \frac{C}{t^2}|f(B(z_0,2r)|$$

where z_0 is taken as in the statement of the lemma. □

(This marks the end of the lecture, the following takes place with the classes on January 24th and January 27th)

We pick up from last time, and use lemma 5, the "crucial lemma", to prove the following

Lemma 6. *If f is qasisymetric on a domain $G \subset \mathbb{R}^2$, then $L_\epsilon^f(z) \in L^p_{loc}(G)$ for $0<p<2$.*

Proof. We appeal to a standard identity, which says

$$\frac{1}{|B|}\int_B L_\epsilon^p dxdy = \frac{1}{|B|}\int_0^\infty pt^{p-1}|\{L_\epsilon > t\}|dt$$

$$= \frac{1}{|B|}(\int_0^x pt^{p-1}|\{L_\epsilon > t\}|dt + \int_x^\infty pt^{p-1}|\{L_\epsilon > t\}|dt)$$

$$\le x^p + \frac{1}{|B|}\int_x^\infty pt^{p-3}C(H)|f(2B)|dt = x^p - C(H)\frac{|f(2B)|}{|B|}\frac{p}{p-2}x^{p-2}$$

where one can choose the x that minimizes the expression to be $(\frac{|f(2B)|}{|B|})^{1/2}$. □

We now enter the discussion about absolute continuity on lines, and show that for almost every (horizontal/vertical) line in G, the map f restricted to said line is AC.

To do this, write f as $u+iv$, and fix $A<B$ to be the real part of vertical lines in G. Since L_ϵ is in $L^1_{loc}(G)$, then for any $A<a<b<B$, and we have

$$|u(b+iy)-u(a+iy)| \le |f(b+iy)-f(a+iy)| \le \int_a^b L_\epsilon(x+iy)dx$$

which is AC for almost every y, by Fubini. So u is AC on $G\cap(\mathbb{R}+iy)$ for almost every y. The argument that shows u is AC on vertical lines are similar, and showing the result for v is identical. Moreover, this also shows that u_x, u_y, v_x, v_y exist almost everywhere, and one should take notice that whenever they exist they are bounded by $L_\epsilon(z)$, whence they are also in L^p_{loc} for $0<p<2$ as well. We now wish to show the following,

$$\text{limsup}_{r\to 0}\frac{\max_{|w|=r}|f(z+w)-f(z)|}{\min_{|w|=r}|f(z+w)-f(z)|} = \frac{|\partial f| + |\overline{\partial} f|}{|\partial f| - |\overline{\partial} f|} \le H$$

, where we know the first equality at points of differentiability where $|\partial f| + |\overline{\partial} f|$ is nonzero, but wish to show the inequality.

Lemma 7. *(Gehring - Lehto) If f is an open map of a domain G in $\mathbb{R}^2$, and if f is partially differentiable almost everywhere in G, then f is totally differentiable almost everywhere.*

Note that the proof we give relies on the fact that we reside in $\mathbb{R}^2$. One can use Stepanoff's result to extend this. Stepanoff's theorem says if $\limsup_{w\to z} \frac{|f(w)-f(z)|}{|z-w|}$ is finite almost everywhere, then f is differentiable almost everywhere.

Proof. By Egorov's theorem, for every $\alpha > 0$ we have

$$\frac{f(z+t)-f(z)}{t} \to f_x, \text{ and } \frac{f(z+it)-f(z)}{t} \to f_y$$

where convergence is uniform, as $t \to 0$, on $U := G \setminus E_\alpha$. Here E_α has measure less than α. Thus the partials f_x and f_y are continuous on U. Denote by U' the points of U that are points of density in both the real and imaginary directions. We claim that f is differentiable at every point in U'. To show this, suppose without loss of generality that the point $z \in U'$ is 0. Now the error term $e(z)$ is

$$e(z) = f(z) - f(0) - xf_x(0) - yf_y(0)$$
$$= f(z) - f(x) - yf_y(x) + f(x) - f(0) - xf_x(0) + y(f_y(x) - f_y(0))$$

using the triangle inequality gives

$$|e(z)| \le |f(z) - f(x) - yf_y(x)| + |f(x) - f(0) - xf_x(0)| + |y(f_y(x) - f_y(0))| \le 3\epsilon|z$$

The bound on the first term uses the fact that $x \in U'$, the bound on the second term comes by the uniform convergence of the partials shown above, while the third term goes to zero because for a small but fixed y in U', $f_y(x) \to f_y(0)$. For fixed ϵ and now for general $x + iy$ in U, choose x', x'', y', y'' in U' such that $x' < x < x''$ and $y' < y < y''$ has

$$(x'' - x') < \epsilon x \text{ and } (y'' - y') < \epsilon y$$

Set $R = [x', x''] \times [y', y'']$. The function $w \mapsto |f(w) - f(0) - xf_x(0) - yf_y(0)|$ satisfies the maximum principle, so the maximum over R is assumed at some $z_1 \in \partial R$. We then have

$$e(z) \le |f(z_1) - f(0) - xf_x(0) - yf_y(0)| \le |e(z_1)| + |x - x_1||f_x(0)| + |y - y_1||f_y(0)|$$
$$\le 3\epsilon|z_1| + |z - z_1|(|f_x(0)| + |f_y(0)|) \le 3\epsilon\frac{|z|}{1-\epsilon} + \frac{\epsilon|z|}{1-\epsilon}(|f_x(0)| + |f_y(0)|)$$
$$\le \frac{\epsilon|z|}{1-\epsilon}(3 + |f_x(0)| + |f_y(0)|)$$

This quantity is bounded by $\epsilon|z|$ multiplied by some constant. □

If f is a homeomorphism, define $\mu_f(A)$ as $|f(A)|$, a Borel measure on G. Set

$$\mu_f'(x) = \lim_{r\to 0} \frac{\mu_f(B(x,r))}{|B(x,r)|}$$

which exists almost everywhere.

Notice that, $\int_A \mu_f'(x)dx \le \mu_f(A)$ and that (i) If f is differentiable at x, and (ii) if $\mu_f'(x)$ exists, then $\mu_f'(x) = Jf(x)$. So we also have $Jf \in L^1_{loc}$.

Exercise 8. Does (i) imply (ii) in this case?