Review: assume $f : G \to \mathbb{R}^2$ is a differmorphism, then it is K quasiconformal if and only if $|\mu_f| \leq k = \frac{K-1}{K+1}$.

Below we go over a few examples of quasiconformal mappings.

Example 0.1. $f(x) = z|z|^{K-1}$. This is the *K-quasiconformal stretching*, which stretches radially.

Example 0.2. Let $f : re^{i\theta} \mapsto re^{i(t+\alpha \log r])}$. This example is also L bi-Lipschitz, and is self similar under dilations (exercise).

The main result we'll tackle today is the following.

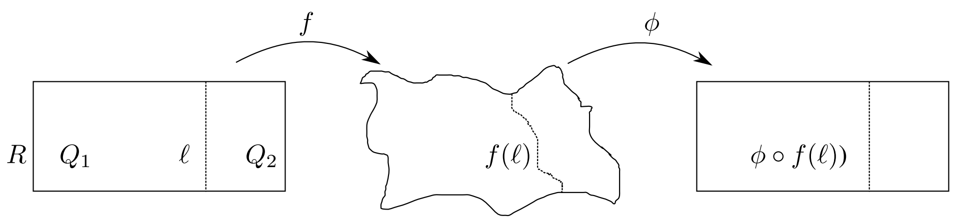
Theorem 0.3. *1-QC maps are conformal.*

To do so we require a lemma. The first part of it we've already proven, but the second part is new.

Lemma 0.4. *If a rectangle R is subdivided by a curve γ, then*

$$M(R) \geq M(Q_1) + M(Q_2)$$

Moreover, equality holds if and only if γ is a straight line.



Proof of Theorem 0.3. We will prove the theorem using the preceeding Lemma. Fix $R \subseteq G$, Let ϕ be the conformal mapping from $f(R)$ back to R that maps the image of the corners of R under f back to the original corners. The moduli of G_1 and G_2 are unchanged, and the lemma says that $\phi \circ f(\gamma)$ is a straight horizontal line, and moreover, that the rectangles it divides R into are the same proportions as before, so in particular $\phi \circ f(\gamma) = \gamma$. Moreover, if we take a horizontal line h, $\phi \circ f(h) = h$ for each hroizontal line. Thus, $\phi \circ f$ is the identity, and so ϕ is the inverse of f. Since ϕ is conformal, f is conformal. □

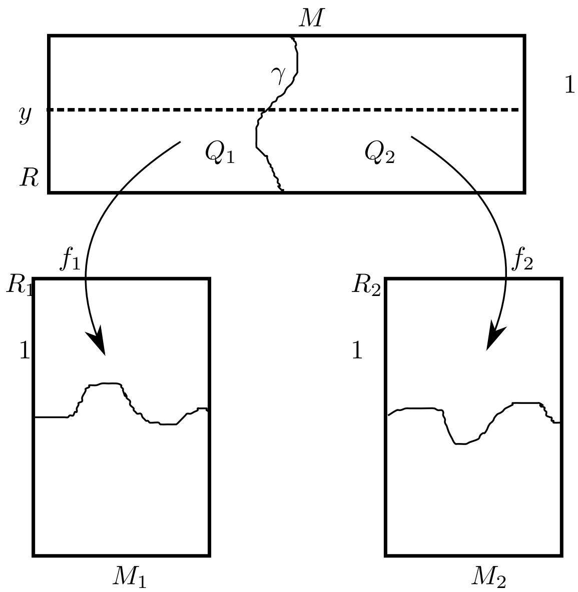
Proof of Lemma. Let R be a 1 by M rectangle.

Let $f_j : Q_j \to R_j$be conformal maps to rectangles R_j of modulus $M(Q_j)$ (so of bases and widths M_j and 1 respectively). Set

$$\rho_j = |f_j'(z)| \text{ on } Q_j.$$

Set

$$\rho = \begin{cases} \rho_1 & \text{on } Q_1 \\ \rho_2 & \text{on } Q_2 \end{cases}$$



Then

$$\iint_R \rho^2 = M_1 + M_2 = M = \iint_R 1$$

thus

$$\iint_R \rho^2 - 1 = 0.$$

Now integrate a horizontal line of height y. Then, since we are taking conformal maps, and y_j denotes two components of y that span both sides of Q_j,

$$\int_{h_y} \rho = \ell(f_1(y)) + \ell(f_2(y)) \geq M_1 + M_2 = M$$

thus integrating over all values of y (from 0 to 1) gives

$$\iint \rho \geq M_1 + M_2 = M$$

Thus

$$\iint (\rho - 1)^2 = \iint (\rho^2 - 1 - 2\rho + 2) = \iint (\rho^2 - 1) - 2\iint (\rho - 1) \leq 0$$

thus $\rho = 1$ a.e., thus f' is constant and f is linear. This implies γ must be a line. $\square$

1 Sobolev Spaces

Recall integration by parts. If $f \in C^1$ on $\mathbb{R}$, and $\phi \in C_0^\infty$, then

$$\int_{\mathbb{R}} f'\phi = -\int_{\mathbb{R}} f\phi'.$$

If instead f is defined on $\mathbb{R}^d$ and $\phi \in C_0^\infty(\mathbb{R}^d)$, then the same holds with partial derivatives:

$$\int_{\mathbb{R}^d} f_x \phi = -\int f \phi_x.$$

So now we can define a sort of artificial derivative of any locally integrable function. That is, we define f_x to be the distribution such that the above equality is satisfied, where we interpret the integral on the left as the distribution f acting on the function ϕ.

Definition 1.1. The function $f \in L^p_{\text{loc}}(G)$ belongs to the *Sobolev space* $W^{1,p}_{\text{loc}}(G)$. If there exists a function $f_1, f_2 \in L^p_{\text{loc}}$ so that for any $\phi \in C_0^\infty(G)$,

$$\int f_1 \phi = -\int f \phi_x$$

and

$$\int f_2 \phi = -\int f \phi_y.$$

The functions f_1 and f_2 are called the *weak derivatives* of f. We let $W^{1,p}(G)$ denote the subspace of $W^{1,p}_{\text{loc}}$ whose functions and their weak derivatives are p integrable. The $W^{1,p}(G)$ norm is defined as

$$||f||_{1,p} = ||f||_p + ||f_1||_p + ||f_2||_p.$$

The theorem we'd like to prove eventually is the following:

Theorem 1.2. *A function f is an orientation preserving K-QC homeomorphism in $\mathbb{R}^2$ (or $G \subseteq \mathbb{R}^2$) if and only if $f \in W^{1,2}_{\text{loc}}$ and $|\bar{\partial} f| \leq k|\partial f|$ a.e.*

Definition 1.3. A real-valued function $u : G \to \mathbb{R}$ is ACL (absolutely continuous on lines) if u is absolutely continous on almost every horizontal and vertical line intersected with G. A function $v : \mathbb{R} \to \mathbb{R}$ is *absolutely continuous* if $v(x) - v(y) = \int_x^y v_1$ for some function $v_1 \in L^1_{\text{loc}}$. Alternatively, $\forall \varepsilon > 0$ there is $\delta > 0$ so that for some collection of intervals I_j,

$$\sum_j |I_j| < \delta \Rightarrow \sum_j |v(I_j)| < \varepsilon.$$

The main feature of ACL functions we will be using is integration by parts.
DAY 2

Recall that we're assuming f is a orientation preserving homeomorphism of $\mathbb{C}$ (or on some subdomain). Our goal is to show that f is K-QC if and only if $f \in W^{1,2}_{\text{loc}}$ and $|\bar{\partial} \leq k|\partial f|$.

Here we give an example that is sort of a warning as to why we have to look at these analytic quantities.

Example 1.4. Recall the Cantor staircase function $u : [0,1] \to [0,1]$. This is increasing, continuous, but is constant on the complement of the Cantor set. Hence, it can't possibly be absolutely continuous.

Now consider the function $f(x, y) = (x+u(x), y)$. Then this is a homeomorphism, and is differentiable almost everywhere. Moreover, we can compute the Beltrami coefficient:

$$\partial f = 1, \quad \bar{\partial} f = 0 \text{ on } (\mathbb{R} \backslash C) \times \mathbb{R}.$$

It is *not* QC since $f \notin W^{1,2}_{\text{loc}}(\mathbb{R}^2)$, or more the point, because it is not ACL.

Lemma 1.5. *If $f : G \to \mathbb{R}$ is continuous, then $f \in W^{1,p}_{\text{loc}}$ for some $p \geq 1$ if and only if f is ACL and the partial derivatives in x and y (which exist a.e.) are p integrable.*

Proof. Let's first start with the forward direction, assume $f \in W^{1,p}_{\text{loc}}(G)$. Fix $R = [0, A] \times [0, B] \subseteq G$. Note that we are given weak derivatives f_1 and f_2. What we want to show is that for every $a \in [0, A]$ and a.e. $y \in [0, B]$, so $(a, y) \in R$, then

$$f(a, y) - f(0, y) = \int f_1(x, y) dx.$$

Observe that by continuity of f, we may assume that $a \in \mathbb{Q}$ and derive the general case using a limiting argument.

Fix a and call $R_b = [0, a] \times [0, b]$. Then

$$\iint_{R_b} f \phi_x = - \iint_{R_b} f_1 \phi$$

for $\phi \in C^\infty_0(R_b)$. Pick $\phi(x, y) = \varphi(x)\psi(y)$, a product of two functions compactly supported in $[0, a]$ and $[0, b]$ respectively. Then the above equality implies that for this ϕ,

$$\iint_{R_b} f \varphi'(x)\psi(y) = - \iint f_1 \varphi(x)\psi(y).$$

By Fubini's theorem, this is the same as

$$\int_0^b \int_0^a f \varphi'(x)\psi(y) dx dy = - \int_0^b \int_0^a f_1 \varphi(x)\psi(y) dx dy$$

which is an absolutely continuous function in the b variable, which implies

$$\int_0^a f(x, b)\varphi'(x) dx = \int_0^a f_1(x, b)\phi(x) dx$$

for a.e. b. Now take φ increasing to 1, this will imply

$$\int_0^a f_1(x, b) dx = f(a, b) - f(0, b)$$

for almost every b.

For the other direction, note that absolute continuity implies we may use integration by parts on almost every line, and then we may use Fubini.

□

Remark 1.6. In fact, it can actually be shown that if f is K quasiconformal, then $f \in W^{1,2+\varepsilon(k)}_{\text{loc}}$, although we won't prove this.

We know that quasiconformal implies quaisymmetric.

Theorem 1.7. *If f is H-quasisymmetric, then $f \in W^{1,2}_{\text{loc}}$ and $|\bar{d}f| \leq K|\partial f|$ with $K = K(H)$.*

Definition 1.8. A homeomorphism $f : (X, d_X) \to (Y, d_Y)$ is η-quasisymmetric for some increasing endomorphism η on $[0, \infty)$ and

$$|x - a| \leq t|x - b| \Rightarrow |f(x) - f(a)| \leq \eta(t)|f(x) - f(b)|.$$

We say that f is weakly H-quasisymmetric if

$$|x - a| \leq |x - b| \Rightarrow |f(x) - f(a)| \leq H|f(x) - f(b)|.$$

(Insert generic picture illustrating this definition with the eggyolk)

Clearly, quasisymmetry implies weak quasisymmetry, although in some scenarios, the converse also holds.

Theorem 1.9. *If $f : X \to Y$ is a homeomorphism, and X, Y are both doubling (that is, there is a constant C so that every ball of radius R in X can be covered by C-many balls of radius $R/2$) and X is connected, then f is quasisymmetric if and only if it is weakly quasisymmetric.*