Lecture 2:

The geometric definition of quasiconformality:

Definition A homeomorphism f of $G \subset \mathbb{R}^2$ is $K-$quasiconformal $(k \geq 1)$ if f is orientation preserving and for any quadrilateral $Q \subset G$, we have

$$\frac{1}{K}M(Q) \leq M(f(Q)) \leq KM(Q) \tag{1}$$

Note that it is an equivalent definition to replace the quadrilateral Q by rectangles with sides parallel to the axes, topological annuli, or the set of curves in the domain G.
The modulus of an annulus is related to the geometry:

Theorem. For every $E, F \subset \mathbb{R}^n$, $n \geq 2$ continua (i.e. compact connected sets), there exists two increasing functions ϕ_0 and ϕ_1 with $\phi_0(\infty) = \infty$ and $\phi_1(0) = 0$ such that

$$\phi_0\left(\frac{\text{dist}(E,F)}{\min\{\text{diam}(E), \text{diam}(F)\}}\right) \leq \frac{1}{M(\Gamma, E, F)} \leq \phi_1\left(\frac{\text{dist}(E,F)}{\min\{\text{diam}(E), \text{diam}(F)\}}\right) \tag{2}$$

Corollary. If f is $K-$quasiconformal in $\mathbb{R}^n$, then $f(\mathbb{R}^n) = \mathbb{R}^n$

Note that if f is a conformal map from the disk to a domain G, and γ: $[0,1) \to G$ is a curve in G such that $\lim\limits_{t\to 1}\gamma(t)$ exists, then $\lim\limits_{t\to 1} f^{-1}(\gamma(t))$ exists.

Lecture 3:

Definition A homeomorphism $f : (X, d_X) \to (Y, d_Y)$ is called $H-$ quasi-symmetric, if for every (x, r), we have

$$H(f, x, r) = \frac{\max_{|y-x|=r} |f(x) - f(y)|}{\min_{|y-x|=r} |f(x) - f(y)|} \tag{3}$$

Definition The linear dilatation of f is $H_f(x) = \overline{\lim_{r \to 0}} H(f, x, r)$

Theorem. If f is a $k-$ quasiconformal homeomorphism of $\mathbb{R}^n$, then f is $H-$quasi-symmetric, with $H := H(k, n)$

In the rest of the lecture, f is a diffeomorphism.
Clever Notations:

Lemma. The following are easy to prove:

- If $f : \mathbb{R}^2 \to \mathbb{R}^2$ linear, then $f(z) = az + b\overline{z}$
- If f is totally differentiable at z_0, then

$$f(z) = f(z_0) + \partial f(z_0)(z - z_0) + \overline{\partial} f(z_0)\overline{(z - z_0)} + o(|z - z_0|)$$

where,

$$\partial f(z_0) = \frac{1}{2} \left(f_x(z_0) - i f_y(z_0) \right)$$

and

$$\overline{\partial} f(z_0) = \frac{1}{2} (f_x(z_0) + i f_y(z_0))$$

with $f = u + iv$, $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$.

Remark:
If f is complex differentiable at $z_0 \iff \overline{\partial} f(z_0) = 0$.
The Jacobian is given by

$$Jf(z_0) = |\partial f(z_0)|^2 - \left|\overline{\partial} f(z_0)\right|^2$$

Then, we have

$$Jf(z_0) > 0 \iff |\partial f(z_0)| > |\overline{\partial} f(z_0)|$$

Example:
Let $f(z) = |z|^2 = z\overline{z}$. then, $\partial f = \overline{z}$ and $\overline{\partial} f = z$

Exercise:
If $g(z, w)$ is an analytic function in both variables, and if $f(z) = g(z, \overline{z})$, then

$$\partial f(z) = g_z(z, \overline{z})$$

and

$$\overline{\partial} f(z) = g_w(z, \overline{z})$$

Definition Complex dilatation of f

$\mu_f(z) = \dfrac{\overline{\partial} f(z)}{\partial f(z)}$ with $|\mu_f| < 1$

Lemma.

$$H_f(z) = \overline{\lim_{r\to 0}} H(f,z,r) = \frac{|\partial f(z)| + |\overline{\partial} f(z)|}{|\partial f(z)| - |\overline{\partial} f(z)|} = \frac{|Df(z)|^2}{Jf(z)} = \frac{1+|\mu(z)|}{1-|\mu(z)|}$$

Proof. We do this for f linear since by the above lemma it is easy consequence to prove it for differentiable functions f.
Let $f(z) = az + b\overline{z}$ with $z = re^{it}$
Then,

$$f(z) = |a|e^{i\alpha} re^{it} + |b|e^{i\beta} re^{-it} = r\left(|a|e^{i(\alpha+t)} + |b|e^{i(\beta-t)}\right)$$

The maximum for the above expression is attained when $\alpha + t = \beta - t$ and hence $t_0 = \dfrac{\beta-\alpha}{2}$
The minimum for the above expression is attained when $\alpha+t = \beta-t+\pi$ and hence $t_1 = \dfrac{\beta-\alpha}{2} + \dfrac{\pi}{2}$
On the other hand, $\mu = \dfrac{\overline{\partial} f}{\partial f} = \dfrac{b}{a}$ □

<u>Exercise</u>:
Prove $\mu_{f\circ g^{-1}}(g(z)) = \dfrac{\mu_f(z) - \mu_g(z)}{1 - \mu_f(z)\overline{\mu_g(z)}} \left(\dfrac{\partial g(z)}{|\partial g(z)|}\right)^2$

Lecture 4:

Theorem. If f is an orientation preserving diffeomorphism, then TFAE:

a- $sup_{z\in G}|\mu_f(z)| = ||\mu||_\infty = k < 1$

b- $\frac{1}{K}M(\Gamma) \leq M(f\Gamma) \leq KM(\Gamma)$ with $K = \frac{1+k}{1-k} \geq 1$

Proof. $1 \implies 2$
Fix $\Gamma \subset G$, and fix ρ admissable for $f(\Gamma)$. Set $\tilde{\rho}(z) := \rho f(z)|Df(z)|$. Then,

$$l_{\tilde{\rho}}(\gamma) = \int_\gamma \tilde{\rho}|dz| = \int_a^b \rho(f(\gamma(t)))|Df(\gamma(t))||\dot{\gamma}(t)|dt \geq \int_a^b \rho(f(\gamma(t)))|f(\dot{\gamma(t)}) \geq \int_{f\circ\gamma} \rho(w)|dw|$$

Then,

$$\int \tilde{\rho}^2 dxdy = \int \rho(f(z))^2|Df(z)|^2 dxdy = \int \rho(f(z))^2.Jf(z)\frac{1+\mu}{1-\mu}dxdy \leq \frac{1+k}{1-k}\int \rho^2(w)dudv$$

Thus, $M(\Gamma) \leq KM(f\Gamma)$.
The inverse inequality is proved just by taking the an admissable ρ in G and pushing it forward, and repeating the calculation.

$2 \implies 1$
Fix $z \in G$ and $r > 0$. Let Q_r be the square of side length r and direction $\frac{1}{2}\arg\mu(z)$. Fix ϵ, and construct a comparison rectangle. Then, $M(f(Q_r)) \geq M(R_{r,\epsilon})$ for $r \leq r_0(\epsilon)$. Letting $\epsilon \to 0$, we have

$$\frac{L}{l} = \frac{|\partial f| + |\overline{\partial} f|}{|\partial f| - |\overline{\partial} f|} = \frac{1+|\mu|}{1-|\mu|} \leq M(f(Q)) \leq KM(Q) = K$$

□

Example:
Consider the radial stretch $f(z) = z.|z|^{k-1}$, $k > 0$
f is a diffeomorphism of $\mathbb{C}\setminus\{0\}$
Write $f(z) = z(z\overline{z})^{\frac{k-1}{2}} = (z)^{\frac{k+1}{2}}(\overline{z})^{\frac{k-1}{2}}$
Then, $\partial f = \frac{k+1}{2}(z)^{\frac{k-1}{2}}(\overline{z})^{\frac{k-1}{2}}$
And, $\partial f = \frac{k-1}{2}(z)^{\frac{k+1}{2}}(\overline{z})^{\frac{k-3}{2}}$
So, $\frac{\overline{\partial} f}{\partial f} = \frac{k-1}{k+1}\frac{z}{\overline{z}}$
$\implies \mu_f = k\frac{z}{\overline{z}}$, so by the theorem, f is quasi-conformal in $\mathbb{C}\setminus\{0\}$

Lemma. If f is quasi-conformal in $G\setminus\{z_0\}$, the it is conformal in G.

Proof. exercise □