1 Circle Packings

Lemma 1.1 (Ring Lemma)**.** *If $\mathbb{D}$ is surrounded by l disks, then $r_j \geq \epsilon_l$ for all j.*

Proof. Assuming the inner radius r_0 is 1, let D_1 be the largest disk. Then $r(D_1) \geq c_l$. If r_2 were small, since $r_3 \leq Cr_2$, and $r_k \leq C^k r_2$, for all k. □

Lemma 1.2. *$f(0) = 0$, $f(1) = 1$, $f : \mathbb{D} \to \mathbb{D}$, K-qc, conformal outside a set E. Then:*

$$\|f - id\|_\infty \leq c_K \sqrt{|E|}.$$

Theorem 1.3 (Hexagonal packing)**.** *If H_n is a circle packing with combinatorics of n generations of hexagonal packing, then:*

$$\frac{r_n}{r_0} = 1 + o(\frac{1}{n}),$$

where r_n is of the first generation, r_0 the root.

sketch. Denote by H_n^0 the "uniform" hexagonal packing (all circles with same radius). Construct a quasi-conformal homeomorphism of $\mathbb{C}$ defined from H_n^0 to H_n as such: map centers to centers, interstices to interstices by Mobius maps, and extend radially.

Extend $F := f \mid_{H_{\frac{n}{2}}}$ to a K-qc map of $\mathbb{C}$, K universal (the fact that K is universal follows from the Ring lemma). Modify F using reflection as we did previously, and obtain a K-qc homeomorphism that is conformal outside a set E. The key point is the following:

$$|E \cap \mathbb{D}| \leq \frac{C}{n^2}.$$

Once we have this, our theorem follows from the previous lemma. □

We can now prove the Circle packing theorem.

Proof. Let G be a triangulation of S^2, embedded in $\mathbb{C}$. Assume that $\infty \notin G$. Every edge $e \in E$ is identified with $\gamma_e : [0, 1] \to \mathbb{C}$. Fix $\epsilon > 0$. Define $G_\epsilon = \hat{\mathbb{C}} \setminus \cup_{e \in E} (\gamma_e(0, \frac{1}{2} - \epsilon) \cup \gamma_e(\frac{1}{2} + \epsilon, 1)). There exists a conformal map f_\epsilon : G_\epsilon \to D_\epsilon$, with $D_\epsilon = \cup_i D_i^\epsilon$. Moreover, we can assume f_ϵ is hydrodynamically normalized. As $\epsilon \to 0$, D_ϵ goes to a circle packing. For this to hold, we need to show that the diameters of the image circles do not degenerate. We already know that the "outer" circles are bounded above and below. Let C_i be a connected component of $\mathbb{C} \setminus G_\epsilon$ and look at $M(\Gamma(C_i, C_j, G_\epsilon), where C_i$ and C_j adjacent.

$$\begin{aligned} M(\Gamma(C_i, C_j, G_\epsilon)) &\geq M(\Gamma(C_{i,r}, C_{j,r}, D_r)) \\ &\geq \frac{1}{2} M(\Gamma(C_{i,r}, C_{j,r}, \mathbb{C})), \end{aligned}$$

by reflection, where $D_r = D(\gamma_e(\frac{[}{1}2), r)$ and $C_{i,r} = C_i \cap D_r$. Hence, $M(\Gamma(C_i, C_j, G_\epsilon))$ goes to ∞ as ϵ goes to 0, and in consequence, $\frac{dist(D_i, D_j)}{min(diam(D_i), diam(D_j))}$ goes to 0 as ϵ goes to 0. Need to show that $diam(D_j)$ does not go to 0 as ϵ goes to 0 (which is already true for outer circles since they're bounded above and below). Suppose we have two discs D_1 and D_2 with radii $\geq \rho > 0$. Assume r_3 goes to 0 (D_3 neighbor of D_1 and D_2. Denote the neighbors of v_1 by v_j, $4 \leq j \leq n$. By an argument similar to the ring lemma, $diam(D_n) \leq C^{n-3} diam(D_3)$, and hence r_j goes to 0 for every j. But $\sum_{i=2}^{n-1} angle(v_i, v_1, v_{i+1}) \geq C$, contradicting $angle(v_i, v_1, v_{i+1})$ goes to 0. □

Corollary 1.4. *If G is any infinite planar graph, then there exists a circle packing with tangency graph G.*

Proof. Pick a vertex v_0, root of G. For each n, G_n= ball of radius n in d_G (combinatorial distance). Circle pack with $D_0 = \mathbb{D}$. From the ring lemma, get upper and lower bounds for radii in terms of d_G. □

Theorem 1.5. *P is a circle packing of S^2, whose tangency graph is a triangulation. If $S^2 \backslash (\cup_{v \in V} D_v \cup \cup_{f \in F} I(f))$ is countable, then any other packing P' with the same graph is a Moebius map.*