Topics in Geometric Function Theory

The following texts that are on hold in the math research library:

    [A] L. Ahlfors, Lectures on quasiconformal mappings

    [AIM] K. Astala, T. Iwaniecz, G. Martin, Elliptic partial differential equations and quasiconformal mappings in the plane

    [D] P. Duren, Univalent functions

    [GM] J. Garnett, D. Marshall, Harmonic Measure

    [H] J. Heinonen, Lectures on Analysis on Metric Spaces

    [LV] O. Lehto, K. Virtanen, Quasiconformal mappings

    [P] C. Pommerenke, Boundary behaviour of conformal maps

Lecture 1 (mostly review):

Definition. *The set of all conformal maps $f : \mathbb{D} \to \mathbb{C}$ that are normalized by $f(0) = 0$ and $f'(0) = 1$ is denoted S. The set of all conformal maps $f : \{|z| > 1\} \to \mathbb{C}$ that have a simple pole of residue 1 at ∞ is denoted Σ.*

Hence every $f \in S$ has a power series representation

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... \quad (z \in \mathbb{D}),$$

and every $f \in \Sigma$ can be written as

$$f(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + ... \quad (|z| > 1).$$

Important examples are the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + ...,$$

which maps $\mathbb{D}$ to $\mathbb{C} \setminus (-\infty, -1/4]$, and

$$f(z) = z + \frac{1}{z}$$

which maps $\Delta = \{|z| > 1\}$ onto $\mathbb{C} \setminus [-2, 2]$.

Koebe Distortion Theorems. *For $f \in S$ and $G = f(\mathbb{D})$,*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2},$$

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3},$$

$$\frac{1}{4}(1-|z|^2)|f'(z)| \leq \text{dist}(f(z), \partial f(\mathbb{D})) \leq (1-|z|^2)|f'(z)|,$$

in particular

$$\frac{1}{4} \leq dist(0, \partial G) \leq 1.$$

The modulus of curve families. Let Γ be a family of curves in $\mathbb{C}$. A typical situation is that E and F are disjoint sets in a domain G, and $\Gamma = \Gamma(E, F, G)$ consists of all curves from E to F in G.

Definition. *A Borel-measurable function $\rho : \mathbb{C} \to [0, \infty]$ is* admissable *for Γ, if for all $\gamma \in \Gamma$*

$$\ell_\rho(\gamma) = \int_\gamma \rho(z)|dz| \geq 1.$$

The modulus *of Γ is*

$$M(\Gamma) = \inf_\rho \iint_\mathbb{C} \rho^2 dxdy,$$

the inf being over all admissable ρ. If there is no such ρ, then $M(\Gamma) = \infty$. The extremal length *of Γ is $\lambda(\Gamma) = 1/M(\Gamma)$.*

Examples: 1. If $G = [0, a] \times [0, b]$ is a rectangle, E, F are the left and right side, $\Gamma = \Gamma(E, F, G)$, then $M(\Gamma) = b/a$.

2. If $G = \{r < |z| < R\}$ is an annulus and G the family of curves joining $\{|z| = r\}$ with $\{|z| = R\}$ in G, then $M(\Gamma) = 2\pi/\log(R/r)$. If Γ is the family of all curves surrounding $\{|z| = r\}$ in G, then $M(\Gamma) = \log(R/r)/2\pi$.

Theorem. *If Γ is a family of curves in a domain G and if f is conformal in G, then*

$$M(f(\Gamma)) = M(\Gamma).$$

Corollary. *If f is conformal in $\mathbb{D}$ and $A = \{t \in [0, 2\pi) : \text{length}(f[0, e^{it})) < \infty\}$, then $\ell(A) = 2\pi$.*

A compact connected set E is *locally connected* if the connected subsets of E form a neighborhood basis, i.e. for every $\epsilon > 0$ there is $\delta > 0$ such that for every $p \in E$ there is a connected subset $F \subset E$ with $D(p, \delta) \cap E \subset F$ and $\text{diam}\, F < \epsilon$.

Theorem. *A conformal map $f : \mathbb{D} \to G$ is continuous in $\overline{\mathbb{D}}$ if and only if ∂G is locally connected.*

A *cut point* of a connected set E is a point p so that $E \setminus \{p\}$ is disconnected.

Theorem. *Let $f : \mathbb{D} \to G$ be conformal. The following are equivalent:*
   (i) f extends to a homeomorphism of $\overline{\mathbb{D}} \to \overline{G}$.
   (ii) ∂G is a simple closed curve (Jordan curve).
   (iii) ∂G is locally connected and has no cut points.

We discussed three different proofs of the following simple fact: Rectangles can be mapped conformally onto each other with corners being mapped to corners if and only if the ratio of the sidelengths coincides. Similarly, annuli are conformally equivalent if and only if the ratios of the radii are the same. One proof is by reflection, one by computing the modulus, and one is by computing a certain winding number. This topological proof has a discrete variant that we will see later.