

## Lecture 10: January 29

Recall that if  $f$  is a homeomorphism on an open subset of  $\mathbb{R}^n$  then  $\int_A Jf(x)dx \leq |f(A)|$ . The strict inequality can hold, for instance with the Cantor function of Lecture 3. The following theorem gives a necessary condition for equality.

**Theorem 10.1** *If  $f$  is a homeomorphism on  $\Omega \subset \mathbb{R}^2$  and  $f \in W^{1,2}(\Omega)$ , then  $\int_A Jf(x)dx = |f(A)|$ .*

**Proof:** It suffices to show the result for rectangles  $R \subset \Omega$  for which  $f(\partial R)$  is absolutely continuous, because all rectangles in  $\Omega$  can be approximated by such  $R$  (since  $f \in W^{1,2}$ ), and then the set of all rectangles generates the Borel algebra.

First assume  $f$  is smooth and write  $f = u + iv$ . Recall Green's Theorem for domains  $D$  with rectifiable boundary and smooth functions  $P$  and  $Q$ :

$$\int_{\partial D} Pdx + Qdy = \iint_D Q_x - P_y dx dy. \quad (10.1)$$

Now if we take  $P = uv_x$  and  $Q = uv_y$ , then by (10.1) we get

$$\iint_R Jf dx dy = \iint_R u_x v_y - u_y v_x dx dy = \int_{\partial R} uv_x dx + uv_y dy = |f(R)|.$$

where the last equality is a standard application of Green's theorem together with the chain rule.

Now for general  $f = u + iv \in W^{1,p}$ , take smooth approximations  $u_n \rightarrow u$  and then  $v_m \rightarrow v$ . ■

**Corollary 10.2** *If  $f$  is a quasiconformal homomorphism then  $Jf > 0$  a.e. and hence  $|\partial f| > 0$  a.e.*

**Proof:** Suppose not. Then there exists  $A \subset \Omega$  with  $|A| > 0$  such that  $Jf = 0$  on  $A$ . Let  $B = f(A)$ , then by Theorem 10.1 we have  $|B| = \int_A Jf(x)dx = 0$ . But then  $f^{-1}$  is a quasiconformal homomorphism taking a null set  $B$  to a set of nonzero measure  $A$ , which is impossible because  $|f^{-1}B| = \int_B J(f^{-1})(x)dx = 0$ . Thus  $Jf > 0$  a.e, and it follows that  $|\partial f| > 0$  a.e. ■

## Lecture 11: January 31

So far we have proved that  $f$  is a homeomorphism in  $\mathbb{R}^2$ , then

$$f \text{ is K-QC} \implies f \text{ is H-QS} \implies f \in W_{loc}^{1,2} \text{ and } |\bar{\partial} f| \leq k|\partial f| \text{ a.e.}$$

where  $K = \frac{1+k}{1-k}$ .

Today we complete the loop by proving that the last statement implies the first.

**Proposition 11.3** *Suppose  $f$  is a homeomorphism on a planar domain  $\Omega$ ,  $f \in W_{loc}^{1,2}$  and  $|\bar{\partial} f| \leq k|\partial f|$  a.e., then  $f$  is  $K$ -QC where  $K = \frac{1+k}{1-k}$ .*

**Proof:** Let  $Q$  be a quadrilateral in  $\Omega$ , and let  $Q' = f(Q)$  be the image of  $Q$ , which is a quadrilateral. Let  $\varphi : Q \rightarrow \mathbb{C}$  be a conformal map taking  $Q$  to a rectangle  $R$  with unit width, and let  $\psi : Q' \rightarrow \mathbb{C}$  be a conformal map taking  $Q'$  to a rectangle  $R'$  with unit width. Let  $M$  and  $M'$  be the heights of  $R$  and  $R'$  respectively. See Figure 11.1.

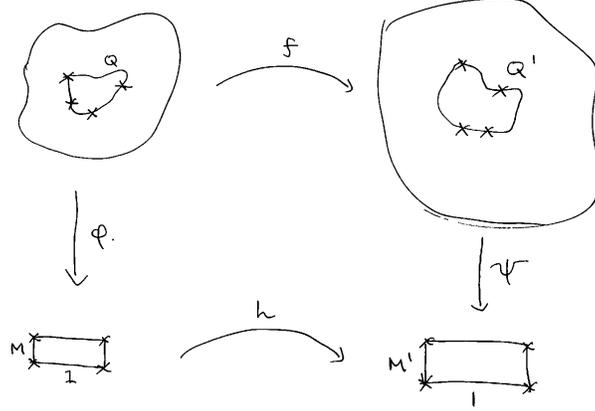


Figure 11.1

Then  $h : R \rightarrow R'$  defined by  $h = \psi \circ f \circ \varphi^{-1}$  is a quasiconformal map.

By the triangle inequality, we have for a.e.  $y$  that

$$\int_0^1 |h_x(x + iy)| dx \geq \left| \int_0^1 h_x(x + iy) dx \right| = |h(1 + iy) - h(0 + iy)| \geq 1,$$

so we get

$$\left( \iint_R |h_x| d\mu \right)^2 = \left( \int_0^M \int_0^1 |h_x(x + iy)| dx dy \right)^2 \geq M^2. \quad (11.2)$$

On the other hand, we have by Cauchy-Schwarz that

$$\begin{aligned} \left( \iint_R |h_x| \right)^2 d\mu &\leq M \iint_R |h_x|^2 d\mu = M \iint_R |h_z + h_{\bar{z}}|^2 d\mu \\ &\leq M \iint_R \frac{|h_z| + |h_{\bar{z}}|}{|h_z| - |h_{\bar{z}}|} (|h_z| - |h_{\bar{z}}|)(|h_z| + |h_{\bar{z}}|) d\mu \\ &= M \iint_R \frac{1 + |\mu_h|}{1 - |\mu_h|} Jhd\mu. \end{aligned}$$

Now  $\frac{1 + |\mu_h|}{1 - |\mu_h|} \leq K$  because  $h$  is a composition of  $f$  with conformal maps (use the chain rule for  $\mu$ ), so we get

$$\left( \iint_R |h_x| \right)^2 \leq MK \int_R Jhd\mu = MK \int_{R'} d\mu = MKM'. \quad (11.3)$$

Now (11.2) and (11.3) show that  $M' \geq M/K$ . Applying the argument to  $h^{-1}$  yields  $M' \leq MK$  and we are done. ■

Here is an example application of the preceding theorem.

**Example 11.4** Let  $Q$  and  $Q'$  be rectangles. By the discussion in Lecture 1, there is no conformal map from  $Q$  to  $Q'$ . But what is the ‘most’ conformal map  $f$  from  $Q$  to  $Q'$ ? Here ‘most’ means the  $f$  that minimizes  $\max_{z \in Q} \frac{1+|\mu_f(z)|}{1-|\mu_f(z)|}$ , or equivalently, minimizes  $\max_{z \in Q} |\mu_f(z)|$ . It is an easy exercise to show that the minimizer is the affine map - since we know how the modulus must change, this already gives a lower bound on  $\max_{z \in Q} \frac{1+|\mu_f(z)|}{1-|\mu_f(z)|}$ .

Every ellipse field is the ellipse field of a quasiconformal map, and this map is unique up to a conformal map.

**Theorem 11.5 (Measurable Riemann Mapping theorem)** If  $\mu : \mathbb{C} \rightarrow \mathbb{D}$  is measurable with  $\|\mu\|_\infty = k < 1$  then there exists a unique orientation preserving homeomorphism  $f : \mathbb{C} \rightarrow \mathbb{C}$  such that  $f$  is quasi-conformal,  $f(0) = 0$ ,  $f(1) = 1$ , and  $\mu_f(z) = \frac{\bar{\partial}f(z)}{\partial f(z)} = \mu(z)$  a.e.

**Proof:** First we prove uniqueness; existence will be done later. Suppose  $f$  and  $g$  satisfy  $\mu_f = \mu$  and  $\mu_g = \mu$  a.e. Then by the chain rule for  $\mu$  we get

$$\mu_{f \circ g^{-1}} = \frac{\mu_f - \mu_g}{1 - \bar{\mu}_g \mu_f} \cdot \left( \frac{\partial g}{|\partial g|} \right)^2 = 0$$

which implies that  $f \circ g^{-1}$  is conformal. If  $f(0) = g(0) = 0$  and  $f(1) = g(1) = 1$  then  $f \circ g^{-1}$  fixes 1 and 0 and so it is the identity map. ■

Now we consider some examples.

1. If  $\mu = c$  for some constant  $c$  then the non-normalized solution is  $f(z) = z + c\bar{z}$ .
2. If  $\mu = c\mathbb{1}_{\mathbb{D}}$  for some constant  $c$  then the solution is

$$f(z) = \begin{cases} z + c/z & \text{if } z \notin \mathbb{D} \\ z + c\bar{z} & \text{if } z \in \mathbb{D}. \end{cases}$$

3. If  $\mu = c\mathbb{1}_Q$  for some constant  $c$  and some square  $Q$ , then the solution is complicated. See Arnaud Chéritat, Beltrami forms, affine surfaces and the Schwarz-Christoffel formula: a worked out example of straightening, arXiv:0811.2601
4. (Joint work with Astala, Saksman, Tao) Divide the unit square into a grid of mesh size  $\delta$ . On each  $\delta \times \delta$  subsquare, define  $\mu$  to be equal to  $1/2$  with probability  $1/2$  and  $-1/2$  with probability  $1/2$ . Then the normalized solution is a random quasiconformal map  $f$ . As  $\delta \rightarrow 0$ , we have that  $f$  is close to the identity with high probability.

**Exercise 11.6** If  $f$  is bi-Lipschitz on  $\mathbb{R}^n$  then  $f$  is quasi-conformal.

**Exercise 11.7 (Open Problem)** Prove that if  $f$  is  $L$ -bi-Lipschitz then  $f = f_1 \circ f_2$  for some  $L_1, L_2$ -bi-Lipschitz maps such that  $L_1 < L$  and  $L_2 < L$ .

If  $f$  is  $K$ -quasi-conformal then we will see below that that the corresponding statement is true. Moreover, for any choice of  $K_1, K_2 > 0$  such that  $K_1 K_2 = K$ , the factors can be chosen in such a way that  $f_j$  is  $K_j$  quasiconformal.

However, is known<sup>1</sup> that some bi-Lipschitz maps cannot be factored in such an optimal way.

<sup>1</sup>Freedman, Michael H., and He, Zheng-Xu. "Factoring the logarithmic spiral.." *Inventiones mathematicae* 92.1 (1988): 129-138.

## Lecture 12: February 5

Today we discuss some consequences of the measurable Riemann mapping theorem.

**Theorem 12.8** *If  $f$  is  $K$ -quasiconformal on  $\mathbb{R}^2$  and  $t \in [0, 1]$ , then there exists a  $K^{1-t}$  quasiconformal  $f_2$  and a  $K^t$  quasiconformal  $f_1$  such that  $f = f_2 \circ f_1$ .*

**Proof:** Recall the hyperbolic distance  $d_h$  on  $\mathbb{D}$ , defined by  $d_h(0, w) = \log \left( \frac{1+|w|}{1-|w|} \right)$  and extending by conformal invariance.

For each  $z$ , let  $u(z)$  the point on the segment  $[0, \mu(z)]$  such that  $d_h(0, u(z)) = td_h(0, \mu(z))$ . That is,

$$\log \left( \frac{1 + |u(z)|}{1 - |u(z)|} \right) = t \log \left( \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \right) \leq t \log K. \quad (12.4)$$

Let  $f_1$  be the orientation preserving homeomorphism such that  $f_1$  is quasi-conformal with Beltrami coefficient  $\mu_{f_1} = u$ . This exists by the measurable Riemann mapping theorem. Applying the exponential map to (12.4) shows that  $f_1$  is  $K^t$ -quasi-conformal.

Now set  $f_2 = f \circ f_1^{-1}$ . We have from the chain rule for  $\mu$  that

$$\mu_{f_2} = \frac{\mu - \mu_{f_1}}{1 - \overline{\mu_{f_1}} \cdot \mu} \cdot \left( \frac{\partial f_1}{|\partial f_1|} \right)^2.$$

Notice that the right hand side is the image of  $\mu$  under a conformal automorphism of  $\mathbb{D}$ , and this particular conformal automorphism also takes  $\mu_{f_1}$  to 0, so by conformal invariance of  $d_h$  we get  $d_h(0, \mu_{f_2}) = d_h(\mu, \mu_{f_1})$ . But  $d_h(\mu, \mu_{f_1}) = (1-t)d(0, \mu) \leq (1-t) \log K$ , so applying the exponential map to both sides shows that  $f_2$  is  $K^{1-t}$ -quasiconformal. ■

The composition of a  $K_1$  quasiconformal map and a  $K_2$  quasiconformal map is a  $K_1 K_2$  quasiconformal map, so Theorem 12.8 says that we can factor optimally. Also, if we set  $d(f, g) = \log K(f \circ g^{-1})$  (the Teichmüller metric), this is a metric on the space of quasiconformal maps. The theorem says that this space is connected and in fact a length space.

Next we consider the extension problem. A homeomorphism  $f : \mathbb{D} \rightarrow \mathbb{D}$  does not always extend to a homeomorphism of the boundary (consider some sort of logarithmic spiral). However, if  $f$  is quasiconformal, then we can say more.

**Theorem 12.9** *If  $f : \mathbb{H} \rightarrow \mathbb{H}$  is quasiconformal then  $f$  extends to a quasiconformal map  $F : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ . Furthermore,  $F|_{\mathbb{R}}$  is real valued and quasisymmetric.*

**Proof:** Let  $\mu : \mathbb{H} \rightarrow \mathbb{D}$  be the Beltrami coefficient of  $f : \mathbb{H} \rightarrow \mathbb{C}$ . We can extend this to all of  $\mathbb{C}$  by reflection:

$$\tilde{\mu}(z) := \begin{cases} \mu(z) & \text{if } z \in \mathbb{H} \\ \mu(\bar{z}) & \text{if } z \in \overline{\mathbb{H}}. \end{cases}$$

Note that we didn't define  $\tilde{\mu}$  on  $\mathbb{R}$ , but that's fine because  $\mathbb{R}$  has measure 0. Now let  $F : \mathbb{C} \rightarrow \mathbb{C}$  be the unique solution to the equation  $\mu_F = \tilde{\mu}$ , normalized in such a way that  $F(0) = 0$  and  $F(1) = 1$  and  $F(\infty) = \infty$ . Let  $G(z) = \overline{F(\bar{z})}$ . Observe that  $G$  is orientation preserving, and  $\mu_G = \mu_F$  by either the chain rule or by drawing

a picture. Hence  $G = F$  by the uniqueness of the measurable Riemann mapping theorem. It follows that for  $x \in \mathbb{R}$ ,  $F(x) = \overline{F(x)}$ , so  $F(\mathbb{R}) \subset \mathbb{R}$ . By either a topological argument or considering the inverse, we get that  $F(\mathbb{R}) = \mathbb{R}$ . Hence either  $F(\mathbb{H}) = \mathbb{H}$  or  $F(\mathbb{H}) = \overline{\mathbb{H}}$ . By considering  $-F$  instead of  $F$ , we can assume wlog that  $F(\mathbb{H}) = \mathbb{H}$ .

Now  $F$  and  $f$  solve the same Beltrami equation in  $\mathbb{H}$ , so  $\Phi := f \circ F^{-1} : \mathbb{H} \rightarrow \mathbb{H}$  is a conformal automorphism of the upper half plane, i.e. a mobius transformation, which extends to a conformal map of  $\hat{\mathbb{C}}$  that fixes  $\mathbb{R}$ . Then  $\tilde{F} = \Phi \circ F$  the desired extension of  $f$ . ■

A related question is the extension of quasimetric maps from  $\mathbb{R} \rightarrow \mathbb{R}$  to quasiconformal  $F : \mathbb{C} \rightarrow \mathbb{C}$ . There are several ways to do this. The Ahlfors-Beurling extension is relatively easy to define, and comes from an integral formula. The Jerison-Kenig extension is also easy to define.

The Douady-Earle extension extends a given quasi-symmetric map  $f : S^1 \rightarrow S^1$  to a quasiconformal map  $\hat{f} : \mathbb{D} \rightarrow \mathbb{D}$ . It is *conformally natural* in the sense that if  $S, T$  are conformal automorphisms of  $\mathbb{D}$  then  $\widehat{S \circ f \circ T} = S \circ \hat{f} \circ T$ .