1 Fri, 2/7/1014

In the next several lectures, we will prove the Measurable Riemann Mapping Theorem. First we will assume $\mu \in C_0^\infty$. In this case f (the solution) is holomorphic near ∞ and has a Laurent series

$$f(z) = az + b + O(\frac{1}{z}).$$

We can assume $a = 1$ and $b = 0$. It follows that

$$\partial f - 1 \in L^p(\mathbb{C}),$$

for all $p > 2$ (by polar coordinate).

Outline.

- We will define Cauchy transform $C\varphi = \frac{1}{\pi}\varphi * \frac{1}{z}$. And show that

$$C\bar{\partial}\varphi = \bar{\partial}C\varphi = \varphi(\text{ for all } \varphi \in C_0^\infty).$$

- Define Beurling transform or complex Hilbert transform $H\varphi = \frac{-1}{\pi}\varphi * \frac{1}{z^2}$. We will see

$$H\varphi = \bar{\partial}C\varphi = C\bar{\partial}\varphi,$$

  and

$$H\bar{\partial}\varphi = C\partial\bar{\partial}\varphi = \partial C\bar{\partial}\varphi = \partial\varphi.$$

  These two transforms are fundamental objects.

- We want

$$\bar{\partial}f = \mu\partial f = \mu(\partial(f(z) - z) + 1)$$
$$\bar{\partial}\varphi = \mu(\partial\varphi + 1) = \mu\partial\varphi + \mu$$

  where $\varphi(z) = f(z) - z$. So

$$\bar{\partial}\varphi = \mu H\bar{\partial}\varphi + \mu.$$

  One needs to find $\bar{\partial}\varphi$ such that

$$(I - \mu H)\bar{\partial}\varphi = \mu.$$

  Solve informally to get

$$\bar{\partial}\varphi = (I - \mu H)^{-1}\mu = \mu + (\mu H)\mu + (\mu H)(\mu H)\mu + \cdots.$$

  Then the solution is $f(z) = \varphi(z) + z = C\bar{\partial}\varphi + z$.

- We will show that H is an isometry: $||H\varphi||_2 = ||\varphi||_2$.

- Cauchy transform is well-defined for smooth functions and L^p, $p > 2$ but not L^2 functions.

- We will use Calderon-Zygmund theorem to show H is bounded on L^p for all $p \geq 2$.

$$||H\varphi||_p \leq C_p||\varphi||_p.$$

(Take on faith or read Ahlfors' book or take Hart Smith's class)

- Need $kC_p < 1$ so that

$$\mu + (\mu H)\mu + (\mu H)(\mu H)\mu + \cdots \text{ converges in } L^p.$$

Riesz-Thorin convexity implies $C_p \to C_2 = 1$ as $p \to 2^+$. So there is p such that $kC_p < 1$.

Remarks.

- Can define Hilbert transform through Fourier transform: $\widehat{H\varphi} = \frac{z}{\bar{z}}\hat{\varphi}$.
- Assume $\mu(z, \lambda)$ depends analytically in λ (for example $\lambda\mu(z)$) then the solution is also analytic in λ.

Theorem 1. *(Cauchy integral formula) If D has rectifiable boundary, $f \in W^{1,1}(\bar{D})$ and f is continuous, then*

$$\text{for } z \in D: \qquad f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\xi)}{\xi - z}\,d\xi - \frac{1}{\pi}\iint_D \frac{\bar{\partial} f(\xi)}{\xi - z}\,dx\,dy.$$

Proof. Green's formula in complex form:

$$\int_{\partial D}\varphi(z)\,dz = \int_{\partial D}\varphi(z)\,dx + i\varphi(z)\,dy = \iint_D (i\varphi_x - \varphi_y)\,dx\,dy = 2i\iint_D \bar{\partial}\varphi\,dx\,dy.$$

Apply this formula to get

$$\int_{\partial(D\setminus D_\varepsilon(z))}\frac{f(\xi)}{\xi - z}\,d\xi = 2i\iint_{D\setminus D_\varepsilon(z)}\frac{\bar{\partial} f}{\xi - z}\,dx\,dy.$$

Let $\varepsilon \to 0^+$ to get the result. □

Remark. The integral in the formula is the principle value (p.v.) integral.

$$\iint_D \frac{\bar{\partial} f(\xi)}{\xi - z}\,dx\,dy = \lim_{\varepsilon\to 0^+}\iint_{D\setminus D_\varepsilon(z)}\frac{\bar{\partial} f(\xi)}{\xi - z}\,dx\,dy.$$

Definition. The Cauchy transform for $\varphi \in L^p_0$, $p > 2$ is

$$C\varphi(z) = \frac{-1}{\pi}\int_{\mathbb{C}}\frac{\varphi(\xi)}{\xi - z}\,dx\,dy \qquad \text{for all } z \in \mathbb{C}.$$

In this case, the integral is in Lebesgue sense (so in p.v. sense)

$$\left|\int_{\mathbb{C}}\varphi(\xi)\frac{1}{\xi - z}\,dx\,dy\right| \leq ||\varphi||_p||\frac{1}{\xi - z}||_{q,B(0,R)}$$

$$\leq C||\varphi||_p,$$

where supp(φ)$\subset B(0,R)$.

Example. When $\varphi = 1_{\mathbb{D}}$, note that $\bar{\partial} C\varphi = \varphi$ then

$$C\varphi = \begin{cases} \bar{z} & \text{in } \mathbb{D}, \\ \frac{1}{z} & \text{in } \mathbb{D}^c. \end{cases}$$

We can compute directly using Green's formula on the domain $\mathbb{D}\backslash D_\varepsilon(z)$ and Cauchy integral formula.

Exercise (*). When $\varphi \in C_0^\infty$, then $C\varphi$ has an linear approximation

$$C\varphi(z) = C\varphi(z_0) + \varphi(z_0)\overline{(z-z_0)} + \frac{-1}{\pi}\left(\iint_B \frac{\varphi(\xi)-\varphi(z_0)}{(\xi - z_0)^2}\,dx\,dy\right)(z-z_0) + o(|z-z_0|),$$

with $B = D(0,R)$ any large ball containing z, z_0 and supp(φ).

Hint. We can show that

$$C\varphi(z) - C\varphi(z_0) = -\frac{1}{\pi}\iint_B \frac{\varphi(z_0)(z-z_0)}{(\xi-z_0)(\xi-z)}\,dx\,dy + \frac{-1}{\pi}\left(\iint_B \frac{\varphi(\xi)-\varphi(z_0)}{(\xi-z_0)^2}\,dx\,dy\right)(z-z_0)$$
$$+\frac{1}{\pi}(z-z_0)^2\iint_B \frac{\varphi(\xi)-\varphi(z_0)}{(\xi-z_0)^2(\xi-z)}\,dx\,dy.$$

Lemma 2. *If $\varphi \in L_0^p$, $p > 2$ then $C\varphi$ exists, is continuous and*

$$|C\varphi(z_1) - C\varphi(z_2)| \leq c_p||\varphi||_p|z_1 - z_2|^{1-2/p}.$$

If $\varphi \in C_0^\infty$ then $\bar{\partial}C\varphi = C\bar{\partial}\varphi = \varphi$.

Proof.

$$|C\varphi(z_1) - C\varphi(z_2)| \leq \frac{1}{\pi}\iint_{\mathbb{C}}\left|\frac{\varphi(\xi)(z_1-z_2)}{(\xi-z_1)(\xi-z_2)}\right|\,dx\,dy$$
$$\leq \frac{|z_1-z_2|}{\pi}||\varphi||_p\left(\iint_{\mathbb{C}}\frac{1}{|\xi-z_1|^q|\xi-z_2|^q}\,dx\,dy\right)^{1/q}$$
$$\leq \frac{|z_1-z_2|}{\pi}||\varphi||_p\left(\iint_{\mathbb{C}}\frac{1}{|\xi|^q|\xi-(z_2-z_1)|^q}\,dx\,dy\right)^{1/q}$$
$$\leq \frac{|z_1-z_2|}{\pi}||\varphi||_p\left(\iint_{\mathbb{C}}\frac{|z_1-z_2|^2}{|z_1-z_2|^{2q}|w|^q|w-1|^q}\,du\,dv\right)^{1/q}$$
$$\leq \frac{|z_1-z_2|^{2/q-1}}{\pi}||\varphi||_p\left(\iint_{\mathbb{C}}\frac{1}{|w|^q|w-1|^q}\,du\,dv\right)^{1/q}.$$

Note that $2/q - 1 = 1 - 2/p$ and that the integral of RHS is bounded since $1 < q < 2$.

In the second statement, one equality comes from the exercise (*), the other comes from Cauchy integral formula.

□

2 Mon, 2/10/2014

Recall two Green's formulae:

(I) $\int_{\partial D} \varphi(z)\, dz = 2i \iint_D \bar{\partial}\varphi \, dx\, dy$.

(II) $\int_{\partial D} \varphi(z)\, d\bar{z} = -2i \iint \partial\varphi \, dx\, dy$.

Definition. Let $\varphi \in C_0^\infty$, we define its Beurling transform (or complex Hilbert transform):

$$H\varphi(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\varphi(\xi)}{(\xi - z)^2}\, dx\, dy.$$

Lemma 3. *$H\varphi(z)$ exists in principle value sense, belongs to C^∞ and*

(i) $H\varphi = \partial C\varphi = C\partial\varphi$.

(ii) $\bar{\partial} H\varphi = H\bar{\partial}\varphi = \partial\varphi$.

(iii) $||H\varphi||_2 = ||\varphi||_2$.

Proof. Fix a large disk $D_R(z_0) \supset \text{supp}\varphi$. Since

$$\iint_{D_R(z_0) - D_\varepsilon(z_0)} \frac{1}{(\xi - z_0)^2}\, dx\, dy = 0$$

by using polar coordinate and $\dfrac{\varphi(\xi) - \varphi(z_0)}{(\xi - z_0)^2}$ is in L^1, the existence of $H(z_0)$ holds in p.v. sense.

(i) Exercise (*) implies that $\partial C\varphi = H\varphi$. Now apply Green's formula (II) to $\xi \mapsto \frac{\varphi(\xi)}{\xi - z}$ on $D_R(z)$,

$$\int_{\partial(D_R(z) - D_\varepsilon(z))} \frac{\varphi(\xi)}{\xi - z}\, d\bar{\xi} = -2i \iint_{D_R - D_\varepsilon} \left(\frac{\partial\varphi(\xi)}{\xi - z} - \frac{\varphi(\xi)}{(\xi - z)^2} \right) dx\, dy.$$

The LHS tends to 0 as $\varepsilon \to 0^+$ since $\varphi = 0$ on large circle $\partial D_R(z)$. The RHS converges to a multiple of $C\partial\varphi - H\varphi$.

(ii) Since $\partial\varphi \in C_0^\infty$, use a lemma of previous lecture,

$$\bar{\partial} H\varphi = \bar{\partial} C\partial\varphi = \partial\varphi,$$

and $H\bar{\partial}\varphi = \partial C\bar{\partial}\varphi = \partial\varphi$.

(iii) It follows from Exercise (*) that $\partial\overline{C\varphi} = \bar{\varphi}$ and $\bar{\partial}\overline{C\varphi} = \overline{H\varphi}$. So

$$\partial(\varphi\,\overline{C\varphi}) = \partial\varphi\,\overline{C\varphi} + |\varphi|^2$$

and

$$\bar{\partial}(H\varphi\,\overline{C\varphi}) = \bar{\partial}H\varphi\,\overline{C\varphi} + |H\varphi|^2 = \partial\varphi\,\overline{C\varphi} + |H\varphi|^2.$$

In order to prove the statement, it suffices to show that

$$\iint_{\mathbb{C}} \partial(\varphi\overline{C\varphi}) = \iint_{\mathbb{C}} \bar{\partial}(H\varphi\,\overline{C\varphi}).$$

The LHS is 0 since for large circle D_R, it follows from the Green's formula (II) that

$$\iint_{D_R} \partial(\varphi\overline{C\varphi})\,dxdy = \frac{-1}{2i}\iint_{\partial D_R} \varphi\overline{C\varphi}\,d\bar{z} = 0.$$

Note: in these lectures, we frequently use the fact that $\varphi = 0$ on large circle ∂D_R.

$$\iint_{D_R} \bar\partial(H\varphi\,\overline{C\varphi})\,dx\,dy = \frac{-1}{2\pi}\int_{\partial D_R} H\varphi\,\overline{C\varphi}\,dz.$$

Note that $H\varphi(z) = O(\frac{1}{|z|^2})$ and $C\varphi(z) = O(\frac{1}{|z|})$. So the integral above tends to 0 as R becomes large.

□

We have discussed $||H\varphi||_{L^p\to L^p} = C_p \to 1$ as $p \to 2^+$.

Proposition 4. *If $\mu \in C_0^\infty$ and $||\mu||_\infty < 1$ then there exists a unique continuous function $f : \mathbb{C} \to \mathbb{C}$ up to an additive constant such that $f \in W^{1,p}_{loc}$ (some $p > 2$), $\mu_f = \mu$ a.e. and $\partial f - 1 \in L^p$. Further f is differentiable, $Jf > 0$ everywhere. And*

$$f(z) = z + C(I - \mu H)^{-1}\mu = z + C\mu(z) + C(\mu H)\mu(z) + C(\mu H)(\mu H)\mu(z) + \cdots$$

Proof. We have discussed that if $C_p||\mu||_\infty < 1$ then $(I - \mu H)^{-1}$ is a bounded operator on L^p, so $(I - \mu H)^{-1}\mu \in L^p_0$ and hence f exists, is continuous, and $\mu_f = \mu$. To prove the differentiability of f, we need some observations (same as in ODE theory). First, the same method proves the existence of solution to

$$\bar\partial g = \mu\partial g + \psi.$$

Recall that there exists f such that $\partial f = a$ and $\bar\partial f = b$ iff $\bar\partial a = \partial b$.

Let $a = e^\varphi$ and $b = \mu e^\varphi$. The above condition is

$$\bar\partial a = e^\varphi\bar\partial\varphi = \mu e^\varphi\partial\varphi + \partial\mu e^\varphi.$$

So $\bar\partial\varphi = \mu\partial\varphi + \partial\mu$. Hence f is differentiable and

$$Jf = (1 - |\mu|^2)|e^{2\varphi}| > 0 \text{ every where.}$$

□

3 Wed, 2/12/2014

Exercise after next lecture. If $||\varphi||_{\mathbb{B}} < 1$ then the function $z \mapsto \int_0^z e^\varphi$ is univalent.

Now we still in the case $\mu \in C_0^\infty$ and

$$f(z) = z + C(I - \mu H)^{-1}\mu.$$

We need to show that f is orientation preserving and a homeomorphism. This is a topological problem.

f is onto. Fix $w \in \mathbb{C}$. Consider a large circle $C_R = \{z : |z| = R\}$. Since $f(z) = z + O(1/|z|)$ near infinity, the curve $f(C_R)$ is close to C_R. Hence the winding number of $f(C_R)$ around w is 1. It follows that there is $z_0 \in D_R(0)$ such that $f(z_0) = w$ since otherwise, we can deform $f(C_R)$ to a point without hitting w. This implies a contradiction.

f is one-to-one. We use the fact that $Jf > 0$ everywhere. It follows that f is locally o.p. homeo. Hence there are finite number of points $z_1, z_2, \cdots, z_n$ that are mapped to w. Consider a small loop γ_j around z_j, $wind(f(\gamma_j), w) = 1$. So

$$1 = wind(C_R, w) = wind(f(C_R), w) = \sum wind(f(\gamma_j), w) = n.$$

The case $\mu \in L^\infty_0$. There exists a sequence $\mu_n \to \mu$ a.e. and μ_n has compact support, and is smooth, uniformly bounded

Let f_n, f be solutions to Beltrami equation w.r.t μ_n, μ. We know that f exists and is continuous since it still makes sense to define $\bar\partial f = (I - \mu H)^{-1}\mu \in L^p_0$ and

$$f(z) = z + C\bar\partial f$$

and many properties of C and H still hold.

We will show that $f_n \to f$ (loc.) uniformly.

$$\begin{aligned} f - f_n &= C\bar\partial(f - f_n) = C(\mu\partial f - \mu_n\partial f_n) \\ &= C[(\mu - \mu_n)\partial f + \mu_n(\partial f - \partial f_n)] \end{aligned}$$

Since $\mu - \mu_n$ has compact support and $\partial f \in L^p_{loc}$,

$$||(\mu - \mu_n)\partial f||_p \to 0$$

and then $||C(\mu - \mu_n)\partial f||_\infty \to 0$.

Note that $\partial f = 1 + \partial C\bar\partial f = 1 + H\mu\partial f$. So

$$\partial(f - f_n) = H(\mu\partial f - \mu_n\partial f_n) = H(\mu - \mu_n)\partial f + H\mu_n(\partial f - \partial f_n).$$

Thus

$$||\partial(f - f_n)||_p \le C_p||(\mu - \mu_n)\partial f||_p + C_p k||\partial f - \partial f_n||_p.$$

Since $C_p k < 1$ we can use a bootstrap argument to show that $||\partial(f - f_n)||_p \to 0$ as $n \to \infty$. It follows that $C(\mu_n(\partial(f - f_n)))$ converges to 0 uniformly and so does $f - f_n$.

Lemma 5. *If f_n is K-qc in $\mathbb{C}$ (also true for $\mathbb{R}^n$) and f_n converges to f locally uniformly then either f is K-qc or constant.*

Proof. It is clear that f is continuous. Suppose that f is not 1-1 and not constant. There exist three different point a, b, c such that $f(a) = f(b) \ne f(c)$. Consider $E = [b, c]$ and F a closed curve passing a and surrounding E. Since f_n is K-qc, mod$(f_n(E), f_n(F), \mathbb{C})$ is uniformly bounded away from 0. However

$$\frac{1}{mod(f_n(E), f_n(F), \mathbb{C})} \asymp \frac{dist(f_n(E), f_n(F))}{\min(\text{diam } (f_n(E), f_n(F)))} \le \frac{|f_n(a) - f_n(b)|}{|f_n(c) - f_n(a)|} \to 0,$$

which implies a contradiction.

Now suppose f is not constant. So f is homeo (on its image). The K-qc conformality of f comes from the continuity of modulus of quadrangles. □

Exercise. The family $\mathcal{F} = \{f : \mathbb{C} \to \mathbb{C} \quad K\text{-qc}, f(0) = 0, f(1) = 1\}$ is normal.

Hint. f is quasi-symmetric.

Now consider the case $\mu \in L^\infty$. Let $\mu_n = 1_{D_n(0)} \cdot \mu$. Let f_n be the solution to Beltrami equation and normalized by $f_n(0) = 0, f_n(1) = 1$. So there exists a subsequence, denoted again by f_n, converging to f, a K-qc that fixes 0 and 1.

Fix n. We see that $f_m \circ f_n^{-1}$ is analytic on $f_n(D_n(0))$ and fixes 0 and 1 for $m \geq n$. So $f \circ f_n^{-1}$ is analytic on $f_n(D_n(0))$ for every n. So

$$\mu_f = \mu_{f_n} = 1_{D_n(0)}\mu \text{ on } D_n(0).$$

4 Fri, 2/14/14

Holomorphic motion. (80-90s). Let $\mu \in L^\infty$. Consider $\mu_f = \lambda \frac{\mu}{||\mu||_\infty}$ for $\lambda \in \mathbb{D}$. Let $f_\lambda(z)$ be the solution to Beltrami equation and normalized at $f(0) = 0$ and $f(1) = 1$. We see that

- Fix λ, $z \mapsto f_\lambda(z)$ is homeomorphic.
- Fix z, $\lambda \mapsto f_\lambda(z)$ is a holomorphic function in $\mathbb{D}$. This can be seen via the proof of Beltrami equation.
- $f_0(z) = z$ for all z.

Definition. A holomorphic motion of a set $A \subset \mathbb{C}$ is an analytic family of injections $f_\cdot(\cdot) : \mathbb{D} \times A \to \mathbb{C}$:

- Fix λ, $z \mapsto f_\lambda(z)$ is homeomorphic.
- Fix z, $\lambda \mapsto f_\lambda(z)$ is a holomorphic function in $\mathbb{D}$.
- $f_0 = id_A$.

Theorem 6. *(1) f extends to a holomorphic motion of $\bar{A}$ and $f_\lambda(\cdot)$ is quasi-symmetric.*

(2) f extends to a holomorphic motion of $\mathbb{C}$ and f_λ is $K = \frac{1+|\lambda|}{1-|\lambda|}$-qc.

Remarks. When f is qc with dilation μ. We can define a holomorphic motion from here. Then let $\lambda = ||\mu||_\infty$ to get f back. So the theorem above says that a QC map is just an instant of a holomorphic motion!

Proof of (1). Fix three points $z_1, z_2, z_3 \in A$, set

$$\varphi(\lambda) = \frac{f_\lambda(z_1) - f_\lambda(z_2)}{f_\lambda(z_1) - f_\lambda(z_3)}.$$

This function is analytic in $\mathbb{D}$, avoids 0 and 1 (and ∞).

Recall the Schottky's theorem, there exists a function $\eta(x, y)$ such that η is increasing in x and y, $\eta(x, 0+) = 0$ and any analytic function φ avoiding 0 and 1

$$|\phi(\lambda)| \leq \eta(|\lambda|, |\varphi(0)|).$$

This theorem is proved either using normal family or universal cover of $\mathbb{C}\backslash\{0,1\}$ and Schwarz's lemma. Now go back to the main theorem. We have

$$\left|\frac{f_\lambda(z_1)-f_\lambda(z_2)}{f_\lambda(z_1)-f_\lambda(z_3)}\right| \le \eta(|\lambda|, \left|\frac{z_1-z_2}{z_1-z_3}\right|).$$

The rest is an exercise.

Ideas of proving (2). Require many papers. The main idea is "holomorphic axiom of choice". Suppose f is a holomorphic motion of n points, we need to find a way to define a motion for the $n+1$-th point.

Applications.

Hausdorff dimension of quasi-circles. It was conjectured that

$$\max_{f \text{ is } k-qc} \dim_H f(\mathbb{R}) = 1 + k^2.$$

Using holomorphic motion, Smirnov proved that

$$\dim_H f(\mathbb{R}) \le 1 + k^2.$$

Astala, Rohde and Schramm showed in the other direction:

$$\max_{f \text{ is } k-qc} \dim_H f(\mathbb{R}) \ge 1 + .69k^2.$$

Julia set. This is the motivation for the discovery of holomorphic motion. See an applet on Istvan Prause's website. One can show that if λ moves around a hyperbolic component of the Mandelbrot set, then the Julia set moves holomorphically.