University of Washington
Complex Analysis - Math 535
S. Rohde Winter 2018

Exercise Set 3

In the first two problems, D and D' are rectangles of sidelengths a, b and a', b' respectively, and $f : D \to D'$ is a conformal map whose homeomorphic extension to $\overline{D}$ sends $a-$sides to $a'-$sides and $b-$sides to $b'-$sides (hence corners to corners).

Problem 1: A continuous function (curve) $\gamma : (0,1) \to \mathbb{C}$ is locally rectifiable if $\gamma[r,s]$ has finite length for all $0 < r < s < 1$. The modulus $M(\Gamma)$ of a set of locally rectifiable curves Γ in D is defined by

$$M(\Gamma) = \inf_{\rho} \int_D \rho^2 dxdy,$$

where the infimum is over all functions $\rho : D \to \mathbb{R}_+$ that are continuous on D and that satisfy

$$\int_\gamma \rho(z)|dz| \geq 1.$$

a) Prove that $M(\Gamma)$ is conformally invariant: If $\Gamma' = f(\Gamma)$ is the set of all curves $f \circ \gamma$, then $M(\Gamma') = M(\Gamma)$.
b) Denote Γ the set of locally rectifiable curves in D that join the $a-$ sides of D. Show that $M(\Gamma) = b/a$ and conclude that $a/b = a'/b'$. Hint: Cauchy-Schwarz.

Problem 2: Assuming that $a/b \neq a'/b'$ and that D and D' "overlap" as sketched in class, consider the winding number of the curve $f(z) - z$ where z ranges over ∂D to obtain a contradiction.

Problem 3. Provide the details to the proof of Runge's theorem in class, regarding the uniform approximation of an integral by Riemann sums.

Problem 4 Provide the details of the proofs of the following two statements:

a) If $U \subset \mathbb{C}$ is open and connected and if $b \in U$, then every rational function with poles in U can be uniformly approximated on $\mathbb{C} \setminus U$ by rational functions with poles only at b.

b) If $U \subset \mathbb{C}$ is open, connected and if there is $R > 0$ with $\{|z| > R\} \subset U$, then every rational function with poles in U can be uniformly approximated on $\mathbb{C} \setminus U$ by polynomials.

Problem 5 Prove the equivalence of the ten characterizations of simply connected domains stated in class.

Due date : Friday, February 9, before class.