Dirichlet-to-Neumann map for Poincaré-Einstein metrics

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Let X be the interior of a compact $(n+1)$-dimensional manifold with boundary $\overline{X}$ and let $[g]$ be a conformal class of Riemannian metrics on ∂X. A Poincaré-Einstein metric with conformal infinity $[g]$ is a Riemannian metric g_+ on X such that $\mathrm{Ric}(g_+) = -ng_+$ and such that $x^2 g_+$ extends at least continuously as a metric to $\overline{X}$ satisfying $x^2 g_+|_{T\partial X} \in [g]$, where x is a defining function for ∂X. A motivating example is the hyperbolic Poincaré metric $h = 4(1-|y|^2)^{-2}\sum(dy^i)^2$ on the ball B^{n+1}, with conformal infinity the conformal class of the usual metric $g^{(0)}$ on the sphere $\mathbb{S}^n$. It was shown in [GL] that if $n \geq 3$ and g is a metric on $\mathbb{S}^n$ which is sufficiently close to $g^{(0)}$, then there is a Poincaré-Einstein metric g_+ near h with conformal infinity $[g]$. Such a metric g_+ is unique up to a diffeomorphism of $\overline{B^{n+1}}$ restricting to the identity on $\mathbb{S}^n$.

If g_+ is a Poincaré-Einstein metric, then a choice of representative metric g in the conformal infinity of g_+ induces for some $\epsilon > 0$ an identification of a neighborhood of ∂X with $\partial X \times [0,\epsilon) \subset \partial X \times \mathbb{R}$ such that in this identification, g_+ takes the geodesic normal form $g_+ = x^{-2}(dx^2 + g_x)$. Here g_x is a 1-parameter family of metrics on ∂X satisfying $g_0 = g$ and x denotes the coordinate in $\mathbb{R}$. If g is C^∞ and n is odd, a boundary regularity theorem asserts that $g_x \in C^\infty(\partial X \times [0,\epsilon))$ (see [A1], [CDLS], [A3], [H]). Moreover, the Taylor expansion of g_x is even to order n: there are ϕ_x, $\psi_x \in C^\infty(\partial X \times [0,\epsilon))$ whose Taylor expansions at $x=0$ contain only even terms such that $g_x = \phi_x + x^n \psi_x$. Of course $g_0 = \phi_0 = g$. One has $\mathrm{tr}_g \psi_0 = 0$ and $\mathrm{div}_g \psi_0 = 0$. It is the case that ψ_0 is locally formally undetermined subject only to these two conditions and the full Taylor expansion of g_x is formally determined in terms of g and ψ_0. The term ψ_0 plays the role of Neumann data for this problem. In the AdS/CFT corresondence, ψ_0 corresponds to the stress-energy tensor of the boundary conformal field theory. For the hyperbolic metric h, one can calculate that $\psi_0 = 0$.

If n is even and g is C^∞, it is shown in [CDLS] that g_x has an expansion involving $\log x$. One can define analogous Neumann data in this case as well, but we assume throughout the rest of this note that n is odd.

Define the Dirichlet-Neumann relation of X to be the set of pairs (g,ψ) such that there is a Poincaré-Einstein metric g_+ on X with conformal infinity $[g]$ and for which ψ is the ψ_0 determined by g. In general, existence and uniqueness fail for Poincaré-Einstein metrics with prescribed conformal infinity. However, by the result of [GL] mentioned above, near $(g^{(0)}, 0)$ the Dirichlet-Neumann relation can be written as the graph of a well-defined Dirichlet-to-Neumann map.

Let $\mathcal{M}^\infty$ denote the space of C^∞ metrics on $\mathbb{S}^n$ and $\mathcal{M}^\infty_0 \subset \mathcal{M}^\infty$ a neighborhood of $g^{(0)}$. Define the Dirichlet-to-Neumann map $\mathcal{N} : \mathcal{M}^\infty_0 \to C^\infty(S^2T^*\mathbb{S}^n)$ as follows. If $g \in \mathcal{M}^\infty_0$, let g_+ be a Poincaré-Einstein metric near h with conformal infinity $[g]$. Write g_+ in the geodesic normal form determined by g and define $\mathcal{N}(g) = \psi_0$. One can show that $\mathcal{N}$ satisfies the following equivariance properties with respect to diffeomorphisms and conformal changes:

$$
\begin{aligned}
&\mathcal{N}(\Phi^* g) = \Phi^* \mathcal{N}(g)\,, \quad \Phi \in \mathrm{Diff}(\mathbb{S}^n) \\
&\mathcal{N}(\Omega^2 g) = \Omega^{2-n} \mathcal{N}(g)\,, \quad 0 < \Omega \in C^\infty(\mathbb{S}^n).
\end{aligned} \tag{1}
$$

The first result below identifies the linearization $d\mathcal{N}_{g^{(0)}}$. For $n \geq 4$, let $\mathcal{W}(g)$ denote the Weyl tensor of the metric g, so that $\mathcal{W} : \mathcal{M}^\infty \to C^\infty(\otimes^4 T^*\mathbb{S}^n)$. Let $W = d\mathcal{W}_{g^{(0)}}$ and let W^* denote the adjoint of W with respect to the L^2 inner product induced by $g^{(0)}$. For $n = 3$, let $\mathcal{C} : \mathcal{M}^\infty \to C^\infty(S^2T^*\mathbb{S}^n)$ denote the Cotton-York tensor of g, normalized by $\mathcal{C}_{ij} = 2\mu_i{}^{kl}\nabla_k P_{jl}$, where $P_{jl} = R_{jl} - \frac{R}{4} g_{jl}$ and μ_{ikl} is the volume form, and set $C = d\mathcal{C}_{g^{(0)}}$. The operator $C : C^\infty(S^2T^*\mathbb{S}^n) \to C^\infty(S^2T^*\mathbb{S}^n)$ is self-adjoint and we set $|C| = \sqrt{C^2}$. Let ∇ denote the covariant derivative and $\Delta = \nabla^*\nabla$ the rough Laplacian with respect to $g^{(0)}$, acting on $C^\infty(S^2T^*\mathbb{S}^n)$. We remark that W^*W commutes with Δ: both are $O(n+1)$-equivariant and they can be simultaneously diagonalized.

Theorem 1. *The linearization of $\mathcal{N}$ at $g^{(0)}$ is given by the following.*

$$
d\mathcal{N}_{g^{(0)}} = \begin{cases} a\, W^*W(\Delta + c_1)\ldots(\Delta + c_m)\sqrt{\Delta + c_{m+1}} & n \geq 5 \\ \frac{1}{3}\,|C| & n = 3\,, \end{cases}
$$

where $m = (n-5)/2$, $a \neq 0$, and $c_i > 0$ for $1 \leq i \leq m+1$.

Theorem 1 is a consequence of the equivariance properties (1). The actions in (1) determine an action on $\mathcal{M}^\infty$ of the semidirect product of the positive C^∞ functions with $\mathrm{Diff}(\mathbb{S}^n)$. The identity component of the isotropy group of $g^{(0)}$ under this action can be identified with the identity component $O_e(n+1,1)$ of the conformal group. Linearizing (1) shows that $d\mathcal{N}_{g^{(0)}}$ is equivariant with respect to two actions of $O_e(n+1,1)$ on $C^\infty(S^2T^*\mathbb{S}^n)$. These actions are principal series representations of $O_e(n+1,1)$ and $d\mathcal{N}_{g^{(0)}}$ is therefore an intertwining operator between two specific principal series representations. A result of [BÓØ] shows that for $n \geq 5$, the space of such intertwining operators is one-dimensional and identifies the spectral decomposition of the intertwining operator. Some computation shows that the operator identified in Theorem 1 is the operator with the prescribed spectrum. For $n = 3$, the space of intertwining operators is 2-dimensional, spanned by C and $|C|$. Consideration of the behavior under orientation reversal shows that $d\mathcal{N}_{g^{(0)}}$ must be a multiple of $|C|$ and evaluation on an example determines the constant.

The properties (1) completely describe the behavior of $\mathcal{N}$ under diffeomorphism and conformal change. This behavior is degenerate: $\mathcal{N}$ collapses the orbit of $g^{(0)}$ to 0. The next result, proved via the implicit function theorem, shows that $\mathcal{N}$ is well-behaved in the transverse directions. Let $\mathcal{M}^{k,\alpha}$ denote the space of $C^{k,\alpha}$ metrics on $\mathbb{S}^n$. Define $\mathcal{T} \subset C^{k,\alpha}(S^2T^*\mathbb{S}^n)$ to be the space of $C^{k,\alpha}$ trace-free, divergence-free tensors with respect to $g^{(0)}$ and let $\mathcal{S}$ be a smooth submanifold of $\mathcal{M}^{k,\alpha}$ near $g^{(0)}$ containing $g^{(0)}$ and tangent to $\mathcal{T}$ at $g^{(0)}$. For example, one choice for $\mathcal{S}$ is the intersection of $g^{(0)} + \mathcal{T}$ with a neighborhood of $g^{(0)}$.

Theorem 2. *For $k > n$, $\mathcal{N}$ extends to a neighborhood of $g^{(0)}$ in $\mathcal{M}^{k,\alpha}$ and $\mathcal{N}|_\mathcal{S} : \mathcal{S} \to C^{k-n,\alpha}(S^2T^*\mathbb{S}^n)$ is a smooth embedding of Banach manifolds.*

Next we describe an application to the LeBrun positive frequency conjecture. First we have the following local unique continuation theorem at infinity for $\pm$ self-dual Poincaré-Einstein metrics.

Theorem 3. *Let X be the interior of a 4-dimensional manifold with boundary $\overline{X}$ and let $U \subset \overline{X}$ be open and connected with $U \cap \partial X \neq \emptyset$. Let g_+ be a Poincaré-Einstein metric on $U \cap X$ with a C^k compactification, where k is fixed but sufficiently large. If g is a metric in the conformal infinity of g_+, then g_+ is $\pm$ self-dual if and only if $3\psi_0 = \pm\mathcal{C}(g)$ on $U \cap \partial X$.*

Theorem 3 is proved by a formal power series analysis of the self-duality equations mentioned in [FG] together with a unique continuation theorem of Mazzeo. Anderson ([A2]) has observed that a globally defined nondegenerate self-dual g_+ satisfies $3\psi_0 = \mathcal{C}(g)$ by consideration of the signature and Gauss-Bonnet formulae.

Let $\mathcal{M}_\pm$ denote the space of metrics g on $\mathbb{S}^n$ near $g^{(0)}$ such that $[g]$ is the conformal infinity of a $\pm$ self-dual Poincaré-Einstein metric. An immediate consequence of Theorem 3 is the following characterization in terms of the Dirichlet-to-Neumann map:

Theorem 4. $\mathcal{M}_\pm = \{g : 3\mathcal{N}(g) = \pm\mathcal{C}(g)\}$

The positive frequency conjecture states that if $\mathcal{S}$ is as above, then $\mathcal{M}_\pm \cap \mathcal{S}$ are smooth submanifolds of $\mathcal{S}$ and $T_{g^{(0)}}\mathcal{S} = T_{g^{(0)}}(\mathcal{M}_+ \cap \mathcal{S}) \oplus T_{g^{(0)}}(\mathcal{M}_- \cap \mathcal{S})$. This was proved by Biquard ([B1]) by deforming the associated twistor spaces. A different proof can be given based on Theorems 1 and 4 by applying the implicit function theorem in a manner similar to that used by Biquard in [B2] for the analogous problem in the asymptotically complex hyperbolic case.

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