Developments in inverse problems since Calderón's foundational paper

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Dedicated to Professor Alberto P. Calderón on the occasion of his 75^{th} birthday.

0 Introduction

In 1980 A. P. Calderón published a short paper entitled "On an inverse boundary value problem" [C I]. This pioneer contribution motivated many developments in inverse problems, in particular in the construction of "complex geometrical optics" solutions of partial differential equations to solve several inverse problems. We survey these developments in this paper. We make emphasis in the new results in the last 5 years since the survey paper [U] was written.

The problem that A. P. Calderón proposed in [C I] is whether it is possible to determine the conductivity of a body by making current and voltage measurements at the boundary. This problem arises in geophysical prospection [Z-K]. Apparently Calderón thought of this problem while working as an engineer in Argentina but he did not publish his results until several decades later. More recently this non-invasive inverse method, also referred in the literature as *Electrical Impedance Tomography*, has been proposed as a possible diagnostic tool and in medical imaging [B-B], [W-F-N]. One concrete clinical application, which seems to be very promising, is in the monitoring of pulmonary edema [I-N-G-Ch], [N-I-C-S-G].

We now describe more precisely the mathematical problem.

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Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary (many of the results we will describe are valid for domains with Lipschitz boundaries). The electrical conductivity of Ω is represented by a bounded and positive function $\gamma(x)$. In the absence of sinks or sources of current the equation for the potential is given by

$$\operatorname{div}(\gamma \nabla u) = 0 \text{ in } \Omega \tag{0.1}$$

since, by Ohm's law, $\gamma \nabla u$ represents the current flux.

Given a potential $f \in H^{\frac{1}{2}}(\partial\Omega)$ on the boundary the induced potential $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\begin{aligned} \operatorname{div}(\gamma \nabla u) &= 0 \text{ in } \Omega, \\ u\big|_{\partial\Omega} &= f. \end{aligned} \tag{0.2}$$

The Dirichlet to Neumann map, or voltage to current map, is given by

$$\Lambda_\gamma(f) = \left(\gamma \frac{\partial u}{\partial \nu}\right)\Big|_{\partial\Omega} \tag{0.3}$$

where ν denotes the unit outer normal to $\partial\Omega$.

The inverse problem is to determine γ knowing Λ_γ. More precisely we want to study properties of the map

$$\gamma \xrightarrow{\Lambda} \Lambda_\gamma. \tag{0.4}$$

Note that $\Lambda_\gamma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is bounded. We can divide this problem into several parts.

a) Injectivity of Λ (identifiability)

b) Continuity of Λ and its inverse if it exists (stability)

c) What is the range of Λ? (characterization problem)

d) Formula to recover γ from Λ_γ (reconstruction)

e) Give an approximate numerical algorithm to find an approximation of the conductivity given a finite number of voltage and current measurements at the boundary (numerical reconstruction).

It is difficult to find a systematic way of prescribing voltage measurements at the boundary to be able to find the conductivity. Calderón took instead a different route.

Using the divergence theorem we have

$$Q_\gamma(f) := \int_\Omega \gamma |\nabla u|^2 dx = \int_{\partial\Omega} \Lambda_\gamma(f) f \, dS \tag{0.5}$$

where dS denotes surface measure and u is the solution of (0.2). In other words $Q_\gamma(f)$ is the quadratic form associated to the linear map $\Lambda_\gamma(f)$, i.e., to know $\Lambda_\gamma(f)$ or $Q_\gamma(f)$ for all $f \in H^{\frac{1}{2}}(\partial\Omega)$ is equivalent. $Q_\gamma(f)$ measures the energy needed to maintain the potential f at the boundary. Calderón's point of view is that if one looks at $Q_\gamma(f)$ the problem is change to find enough solutions $u \in H^1(\Omega)$ of the equation (0.1) in oder to find γ in the interior. We will explain this approach further in the next section where we study the linearization of the map

$$\gamma \xrightarrow{Q} Q_\gamma \tag{0.6}$$

Here we consider Q_γ as the bilinear form associated to the quadratic form (0.5).

In section 1 we describe Calderón's paper and how he used complex exponentials to prove that the linearization of (0.6) is injective at constant conductivities. He also gave an approximation formula to reconstruct a conductivity which is, a priori, close to a constant conductivity.

In section 2 we describe the construction by Sylvester and Uhlmann [S-U I,II] of complex geometrical optics solutions for the Schrödinger equation associated to a bounded potential. These solutions behave like Calderón's complex exponential solutions for large complex frequencies. In section 3 we use these solutions to prove, in dimension $n \geq 3$ a global identifiability result [S-U, I], stability estimates [Al I] and a reconstruction method for the inverse problem [N I], [No]. We also describe an extension of the identifiability result to non-linear conductivities [Su I].

In section 4 we consider the two dimensional case. In particular we follow recent work of Brown and Uhlmann [B-U] to improve the regularity result in A. Nachman's result [N II]. In turn the [B-U] paper relies in work of Beals and Coifman [B-C III] and L. Sung [Sung] in inverse scattering for a class of first order systems in two dimensions.

In section 5 we consider other inverse boundary value problems arising in applications. A common feature of these problems is that they can be reduced to consider first order scalar and systems perturbations of the Laplacian. In the scalar case we consider an inverse boundary value problem for the Schrödinger equation in the presence of a magnetic potential. We also consider an inverse boundary value problem for the elasticity system. The problem is to determine the elastic parameters of an elastic body by making displacements and traction measurements at the boundary. In section 6 we give a general method, due to Nakamura and Uhlmann [N-U I], to construct the complex geometrical optics solutions in this case and a method, due to Tolmasky [To], to construct these solutions for first order perturbations of the Laplacian with less regular conductivities.

Finally we consider in section 7 the case of anisotropic conductivities, i.e. the conductivity depends also of direction. In particular we outline recent progress in the study of the quasilinear case [Su-U III].

1 Calderón's paper

Calderón proved in [C I] that the map Q is analytic. The Fréchet derivative of Q at $\gamma = \gamma_0$ in the direction h is given by

$$dQ|_{\gamma=\gamma_0}(h)(f,g) = \int_\Omega h\nabla u \cdot \nabla v \, dx \tag{1.1}$$

where $u, v \in H^1(\Omega)$ solve

$$\begin{cases} \operatorname{div}(\gamma_0 \nabla u) = \operatorname{div}(\gamma_0 \nabla v) = 0 \text{ in } \Omega \\ u\Big|_{\partial\Omega} = f \in H^{\frac{1}{2}}(\partial\Omega), \quad v\Big|_{\partial\Omega} = g \in H^{\frac{1}{2}}(\partial\Omega). \end{cases} \tag{1.2}$$

So the linearized map is injective if the products of $H^1(\Omega)$ solutions of $\operatorname{div}(\gamma_0 \nabla u) = 0$ is dense in, say, $L^2(\Omega)$.

Calderón proved injectivity of the linearized map in the case $\gamma_0 =$ constant, which we assume for simplicity to be the constant function 1. The question is reduced to whether the product of gradients of harmonic functions is dense in, say, $L^2(\Omega)$.

Calderón took the following harmonic functions

$$u = e^{x\cdot\rho}, \quad v = e^{-x\cdot\overline{\rho}} \tag{1.3}$$

where $\rho \in \mathbb{C}^n$ with

$$\rho \cdot \rho = 0 \tag{1.4}$$

We remark that the condition (1.4) is equivalent to the following

$$\rho = \frac{\eta + ik}{2}, \eta, k \in \mathbb{R}^n \tag{1.5}$$
$$|\eta| = |k|, \eta \cdot k = 0$$

Then plugging the solutions (1.3) into (1.1) we obtain if $dQ|_{\gamma_0=1}(h) = 0$

$$|k|^2(\chi_\Omega h)^\wedge(k) = 0 \qquad \forall\, k \in \mathbb{R}^n$$

where χ_Ω denotes the characteristic function of Ω. Then we easily conclude that $h = 0$. However one cannot apply the implicit function theorem to conclude that γ is invertible near a constant since conditions on the range of Q that would allow use of the implicit function theorem are either false or not known.

Calderón also observed that using the solutions (1.3) one can find an approximation for the conductivity γ if

$$\gamma = 1 + h \tag{1.6}$$

and h small enough in L^∞ norm.

We are given

$$G_\gamma = Q_\gamma\left(e^{x\cdot\rho}\Big|_{\partial\Omega}, e^{-x\cdot\overline{\rho}}\Big|_{\partial\Omega}\right)$$

with $\rho \in \mathbb{C}^n$ as in (**??**). Now

$$\begin{aligned} G_\gamma = &\int_\Omega (1+h)\nabla u\cdot\nabla v\,dx \\ &+ \int_\Omega h(\nabla\delta u\cdot\nabla v + \nabla u\cdot\nabla\delta v)\,dx \\ &+ \int_\Omega (1+h)\nabla\delta u\cdot\nabla\delta v\,dx \end{aligned} \tag{1.7}$$

with u, v as in (1.3) and

$$\begin{aligned} \operatorname{div}(\gamma\nabla(u+\delta u)) &= \operatorname{div}(\gamma\nabla(v+\delta v)) = 0 \text{ in } \Omega \\ \delta u\Big|_{\partial\Omega} &= \delta v\Big|_{\partial\Omega} = 0. \end{aligned} \tag{1.8}$$

Now standard elliptic estimates applied to (1.8) show that

$$\|\nabla\delta u\|_{L^2(\Omega)}, \quad \|\nabla\delta v\|_{L^2(\Omega)} \le C\|h\|_{L^\infty(\Omega)}|k|e^{\frac{1}{2}r|k|} \tag{1.9}$$

for some $C > 0$ where r denotes the radius of the smallest ball containing Ω.

Now plugging u, v into (**??**) we obtain

$$\widehat{\chi_\Omega\gamma}(k) = -2\frac{G_\gamma}{|k|^2} + R(k) = \widehat{F}(k) + R(k) \tag{1.10}$$

where F is determined by G_γ and therefore known. Using (1.9), we can show that $R(k)$ satisfies the estimate

$$|R(k)| \le C\|h\|^2_{L^\infty(\Omega)} e^{r|k|} \tag{1.11}$$

In other words we know $\widehat{\chi_\Omega\gamma}(k)$ up to a term that is small for k small enough. More precisely, let $1 < \alpha < 2$. Then for

$$|k| \le \frac{2-\alpha}{r} \log \frac{1}{\|h\|_{L^\infty}} =: \sigma \tag{1.12}$$

we have that

$$|R(k)| \le C\|h\|^\alpha_{L^\infty(\Omega)} \tag{1.13}$$

for some $C > 0$.

We take $\widehat{\eta}$ a C^∞ cut-off so that $\widehat{\eta}(0) = 1$, $\text{supp}\widehat{\eta}(k) \subset \{k \in \mathbb{R}^n, |k| \le 1\}$ and $\eta_\sigma(\chi) = \sigma^n \eta(\sigma x)$. Then we obtain

$$\widehat{\chi_\Omega\gamma}(k)\widehat{\eta}\left(\frac{k}{\sigma}\right) = \frac{-2G_\gamma\gamma}{|k|^2}\widehat{\eta}\left(\frac{k}{\sigma}\right) + R(k)\widehat{\eta}\left(\frac{k}{\sigma}\right)$$

Using this we get the following estimate

$$|\rho(x)| \le C\|h\|^\alpha_{L^\infty(\Omega)} \left[\log \frac{1}{\|h\|_{L^\infty(\Omega)}}\right]^n \tag{1.14}$$

where $\rho(x) = (\chi_\Omega\gamma * \eta_\sigma)(x) - (F * \eta_\sigma)(x)$. Formula (1.14) gives then an approximation to the smoothed out conductivity, $\chi_\Omega\gamma * \eta_\sigma$, for h sufficiently small.

This approximation estimate of Calderón and modifications of it have been tried out numerically ([Id-Ie]) .

This estimate uses the harmonic exponentials for low frequencies. In the next section we consider high (complex) frequency solutions of the conductivity equation

$$L_\gamma = \text{div}(\gamma\nabla u) = 0$$

2 Complex geometrical optics for the Schrödinger equation

Let $\gamma \in C^2(\mathbb{R}^n)$, γ strictly positive in $\mathbb{R}^n$ and $\gamma = 1$ for $|x| \ge R$ some $R > 0$. Let $L_\gamma u = \text{div}(\gamma\nabla u)$. Then we have

$$\gamma^{-\frac{1}{2}} L_\gamma(\gamma^{-\frac{1}{2}}) = \Delta - q \tag{2.1}$$

where

$$q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}. \tag{2.2}$$

Therefore, to construct solutions of $L_\gamma u = 0$ in $\mathbb{R}^n$ it is enough to construct solutions of the Schrödinger equation $(\Delta - q)u = 0$ with q of the form (2.2). The next result proven in [S-U, I, II] states the existence of complex geometrical optics solutions for the Schrödinger equation associated to any bounded and compactly supported potential.

Theorem 2.1 *Let $q \in L^\infty(\mathbb{R}^n)$, $n \geq 2$, with $q(x) = 0$ for $|x| \geq R > 0$. Let $-1 < \delta < 0$. There exists $\epsilon(\delta)$ and such that for every $\rho \in \mathbb{C}^n$ satisfying*

$$\rho \cdot \rho = 0$$

and

$$\frac{\|(1+|x|^2)^{1/2} q\|_{L^\infty(\mathbb{R}^n)} + 1}{|\rho|} \leq \epsilon$$

there exists a unique solution to

$$(\Delta - q)u = 0$$

of the form

$$u = e^{x\cdot\rho}(1 + \psi_q(x,\rho)) \tag{2.3}$$

with $\psi_q(\cdot,\rho) \in L^2_\delta(\mathbb{R}^n)$. Moreover $\psi_q(\cdot,\rho) \in H^2_\delta(\mathbb{R}^n)$ and for $0 \leq s \leq 1$ there exists $C = C(n,s,\delta) > 0$ such that

$$\|\psi_q(\cdot,\rho)\|_{H^s_\delta} \leq \frac{C}{|\rho|^{1-s}} \tag{2.4}$$

Here

$$L^2_\delta(\mathbb{R}^n) = \{f; \int (1+|x|^2)^\delta |f(x)|^2 dx < \infty\}$$

with the norm given by $\|f\|^2_{L^2_\delta} = \int (1+|x|^2)^\delta |f(x)|^2 dx$ and $H^m_\delta(\mathbb{R}^n)$ denotes the corresponding Sobolev space. Note that for large $|\rho|$ these solutions behave like Calderón's exponential solutions. The equation for ψ_q is given by

$$(\Delta + 2\rho\cdot\nabla)\psi_q = q(1+\psi_q). \tag{2.5}$$

The equation (2.5) is solved by constructing an inverse for $(\Delta + 2\rho\cdot\nabla)$ and solving the integral equation

$$\psi_q = (\Delta + 2\rho\cdot\nabla)^{-1}(q(1+\psi_q)). \tag{2.6}$$

Lemma 2.1 *Let* $-1 < \delta < 0, \quad 0 \le s \le 1$. *Let* $\rho \in \mathbb{C}^n - 0$, $\rho \cdot \rho = 0$. *Let* $f \in L^2_{\delta+1}(\mathbb{R}^n)$. *Then there exists a unique solution* $u_\rho \in L^2_\delta(\mathbb{R}^n)$ *of the equation*

$$\Delta_\rho u_\rho := (\Delta + 2\rho \cdot \nabla) u_\rho = f. \tag{2.7}$$

Moreover $u_\rho \in H^2_\delta(\mathbb{R}^n)$ *and*

$$\|u_\rho\|_{H^s_\delta(\mathbb{R}^n)} \le \frac{C_{s,\delta}\|f\|_{L^2_{\delta+1}}}{|\rho|^{s-1}}$$

for $0 \le s \le 1$ *and for some constant* $C_{s,\delta} > 0$.

The integral equation (2.5) can then be solved in $L^2_\delta(\mathbb{R}^n)$ for large $|\rho|$ since

$$(I - (\Delta + 2\rho \cdot \nabla)^{-1} q)\psi_q = (\Delta + 2\rho \cdot \nabla)^{-1} q$$

and $\|(\Delta + 2\rho \cdot \nabla)^{-1} q\|_{L^2_\delta \to L^2_\delta} \le \frac{C}{|\rho|}$ for some $C > 0$ where $\| \quad \|_{L^2_\delta \to L^2_\delta}$ denotes the operator norm between $L^2_\delta(\mathbb{R}^n)$ and $L^2_\delta(\mathbb{R}^n)$. We will not give details of the proof of Lemma 2.1 here. We refer to the papers [S-U I, II]. We describe the underlying ideas in the case $n \ge 3$.

The point is that the operator $\Delta_\rho = \Delta + 2\rho \cdot \nabla$ has a symbol $-|\xi|^2 + 2i\rho \cdot \xi$ which is jointly homogeneous of degree 2 in (ξ, ρ). Since we want to look at the behavior of Δ_ρ in ρ we consider ρ as another dual variable (this will be made more precise in section 6).

Now the characteristic variety of Δ_ρ in ξ-space for every ρ is a codimension two real submanifold. One simple example that exhibits both behaviors is the equation $|\rho|(\partial x_1 + i\partial x_2)$ in $\mathbb{R}^n$. We have that the "principal symbol" of $|\rho|(\partial x_1 + i\partial x_2)$ is homogeneous of degree two in (ξ, ρ) and its characteristic variety has codimension two. The point then is that Δ_ρ is microlocally equivalent to $|\rho|(\partial x_1 + i\partial x_2)$ and the estimates follow from the Nirenberg-Walker [N-W] estimates for the $\overline{\partial}$ equation in two dimensions. Namely in [N-W] it is proved the following.

Lemma 2.2 *Let* $n = 2$. *Let* $-1 < \delta < 0$. *Let* $L = \partial$ *or* $\overline{\partial}$. *Then given* $f \in L^2_{\delta+1}(\mathbb{R}^2)$ *there exists a unique* $u \in L^2_\delta(\mathbb{R}^2)$ *so that*

$$Lu = f.$$

Moreover $\|u\|_{L^2_\delta} \le C\|f\|_{L^2_{\delta+1}}$ *for some* $C = C(\delta) > 0$.

Now if we apply this result to $|\rho|\frac{(\partial x_1 + i\partial x_2)}{2}$ with the variables $x_3, \dots, x_n$ as parameters we get lemma 2.1 for $s = 0$ since

$$\begin{aligned}
\|u\|^2_{L^2_\delta} &= \int_{\mathbb{R}^n} (1 + |x|^2)^\delta |u(x)|^2 dx \le \int_{\mathbb{R}^n} (1 + |x_1|^2 + |x_2|^2)^\delta |u(x)|^2 dx \\
&\le \frac{C}{|\rho|} \int_{\mathbb{R}^n} (1 + |x_1|^2 + |x_2|^2)^{\delta+1} |f(x)|^2 dx \\
&\le \frac{C}{|\rho|} \int_{\mathbb{R}^n} (1 + |x|^2)^{\delta+1} |f(x)|^2 dx.
\end{aligned}$$

We mention here the following extension due to R. Brown ([Br I]) of Lemma 2.1 to Besov spaces.

Lemma 2.3 *Let $\rho \in \mathbb{C}^n - 0$, $n \geq 3$, satisfying $\rho \cdot \rho = 0$. Then for $-1 < \delta < 0$ and $0 \leq s \leq \frac{1}{2}$ we have that*

$$\|\Delta_\rho^{-1} f\|_{B_{2,2}^{s,\delta}} \leq \frac{C}{|\rho|^{1-2s}} \|f\|_{B_{2,2}^{-s,\delta+1}}$$

where $C = C(n, s, \delta)$.

Here $B_{p,q}^{s,\delta} = \{f : (1 + |x|^2)^{\frac{\delta}{2}} f \in B_{p,q}^s\}$ with the norm

$$\|f\|_{B_{p,q}^{s,\delta}} = \|(1 + |x|^2)^{\frac{\delta}{2}} f\|_{B_{p,q}^s}$$

and $B_{p,q}^s$ denotes the standard Besov space, i.e., $f \in B_{p,q}^s$ if and only if

$$\|f\|_{L^p} + (\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx)^{q/p} h^{-n-sq} dh)^{\frac{1}{q}} \tag{2.8}$$

is finite and (2.8) gives a norm.

We shall discuss the proof of Lemma 2.1 in the two dimensional case in section 4 together with and an extension of the estimates to weighted L^p spaces.

In Theorem 3.1 one assumes that $\gamma \in C^2(\overline{\Omega})$ in order to have $q \in L^\infty(\Omega)$. Brown [Br I] showed that one can relax the smoothness assumption on the conductivity further. Let γ be a bounded function on $\mathbb{R}^n$ strictly positive and γ equal 1 for $|x| > M$ and for some $0 < s < 1$

$$\|\nabla \gamma\|_{B_{\infty,2}^{1-s}} \leq M. \tag{2.9}$$

Let $u \in C^\infty(\mathbb{R}^n)$. We denote by $m_q(u)$ be the distribution defined by

$$m_q(u)(\varphi) = -\int_{\mathbb{R}^n} \nabla\sqrt{\gamma} \cdot \nabla \left(\frac{1}{\sqrt{\gamma}} u\varphi\right) dx, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^n).$$

Note that if $\gamma \in C^2(\mathbb{R}^n)$ then

$$m_q(u) = qu \text{ with } q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}.$$

In [Br I] it was proven that the map m_q is bounded between certain Besov spaces. More precisely we have

$$\|m_q(u)\|_{B_{2,2}^{-s,\delta+1}} \leq C\|u\|_{B_{2,2}^{s,\delta}} \tag{2.10}$$

for $-1 < \delta < 0$, $0 < s < 1$. Combining Lemma 2.3 and (2.10) one concludes

Theorem 2.2 ([Br I]) *Let γ be a bounded function in $\mathbb{R}^n$ strictly positive and one outside a large ball. Let $\rho \in \mathbb{C}^n$ satisfy $\rho \cdot \rho = 0$. Let $0 < s < \frac{1}{2}$ and $-1 < \delta < 0$. Let $f \in B_{2,2}^{-s,\delta+1}$. Then $\exists\, R > 0$ such that for $|\rho| > R$ there exists a unique solution $\psi \in B_{2,2}^{s,\delta}$ to*

$$\Delta\psi + 2\rho \cdot \nabla\psi - m_q(\psi) = f.$$

Furthermore we have for some $C = C(n, s, \delta, M) > 0$

$$\|\psi\|_{B_{2,2}^{s,\delta}} \leq \frac{C}{|\rho|^{1-2s}} \|f\|_{B_{2,2}^{-s,\delta+1}}$$

3 The inverse conductivity problem in $n \geq 3$

The identifiability question was resolved in [S-U I] for smooth enough conductivities. The result is

Theorem 3.1 *Let $\gamma_i \in C^2(\overline{\Omega})$, γ_i strictly positive, $i = 1, 2$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then $\gamma_1 = \gamma_2$ in $\overline{\Omega}$.*

In dimension $n \geq 3$ this result is a consequence of a more general result. Let $q \in L^\infty(\Omega)$. We define the Cauchy data as the set

(3.1)
$$\mathcal{C}_q = \left\{ \left(u\Big|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \right) \right\}, \quad \text{where } u \in H^1(\Omega)$$

is a solution of

(3.2)
$$(\Delta - q)u = 0 \text{ in } \Omega.$$

We have that $\mathcal{C}_q \subseteq H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$. If 0 is not a Dirichlet eigenvalue of $\Delta - q$, then in fact $\mathcal{C}_q$ is a graph, namely

$$\mathcal{C}_q = \{(f, \Lambda_q(f)) \in H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)\}$$

where $\Lambda_q(f) = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$ with $u \in H^1(\Omega)$ the solution of

$$\begin{aligned} (\Delta - q)u &= 0 \text{ in } \Omega \\ u\Big|_{\partial\Omega} &= f. \end{aligned}$$

Λ_q is the *Dirichlet to Neumann* map in this case.

Theorem 3.2 *Let $q_i \in L^\infty(\Omega)$, $i = 1,2$. Assume $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$, then $q_1 = q_2$.*

We now show that Theorem 3.2 implies Theorem 3.1.

Using (2.1) we have that

$$\mathcal{C}_{q_i} = \left\{ \left(f, \left(\frac{1}{2}\gamma_i^{-\frac{1}{2}}\Big|_{\partial\Omega} \frac{\partial \gamma_i}{\partial \nu}\Big|_{\partial\Omega} \right) f + \gamma_i^{-\frac{1}{2}}\Big|_{\partial\Omega} \Lambda_{\gamma_i}\left(\gamma^{-\frac{1}{2}}\Big|_{\partial\Omega} f\right)\right), \quad f \in H^{\frac{1}{2}}(\partial\Omega) \right\}.$$

Then we conclude $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ since we have the following result due to Kohn and Vogelius ([K-V I]).

Theorem 3.3 *Let $\gamma_i \in C^1(\overline{\Omega})$ and strictly positive. Assume $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then*

$$\partial^\alpha \gamma_1\Big|_{\partial\Omega} = \partial^\alpha \gamma_2\Big|_{\partial\Omega}, \qquad |\alpha| \le 1.$$

Remark 3.1. In fact Kohn and Vogelius proved that if $\gamma_i \in C^\infty(\overline{\Omega})$, γ_i strictly positive then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies that

$$\partial^\alpha \gamma_1\Big|_{\partial\Omega} = \partial^\alpha \gamma_2\Big|_{\partial\Omega} \qquad \forall\ \alpha.$$

This settled the identifiability question in the real-analytic category. They extended the identifiability result to piecewise real-analytic conductivities in [K-V II].

Proof of Theorem 3.2. Let $u_i \in H^1(\Omega)$ be a solution of

$$(\Delta - q_i)u_i = 0 \text{ in } \Omega, \qquad i = 1,2.$$

Then using the divergence theorem we have that

$$\int_\Omega (q_1 - q_2)u_1 u_2 dx = \int_{\partial\Omega} \left(\frac{\partial u_1}{\partial \nu} u_2 - u_1 \frac{\partial u_2}{\partial \nu} \right) dS. \tag{3.3}$$

Now it is easy to prove that if $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ then the LHS of (3.3) is zero.

Now we extend $q_i = 0$ in Ω^c. We take solutions of $(\Delta - q_i)u_i = 0$ in $\mathbb{R}^n$ of the form

$$u_i = e^{x\cdot\rho_i}(1 + \psi_{q_i}(x, \rho_i)), \qquad i = 1,2 \tag{3.4}$$

with $|\rho_i|$ large, $i = 1,2$, with

$$\rho_1 = \frac{\eta}{2} + i\left(\frac{k+l}{2}\right) \tag{3.5}$$

$$\rho_2 = -\frac{\eta}{2} + i\left(\frac{k-l}{2}\right)$$

and $\eta, k, l \in \mathbb{R}^n$ such that

$$\begin{aligned} \eta \cdot k = k \cdot l = \eta \cdot l = 0 \\ |\eta|^2 = |k|^2 + |l|^2. \end{aligned} \tag{3.6}$$

Condition (**??**) guarantees that $\rho_i \cdot \rho_i = 0$, $i = 2$. Replacing (3.4) into

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0 \tag{3.7}$$

we conclude

$$\widehat{(q_1 - q_2)}(-k) = -\int_\Omega e^{ix\cdot k}(q_1 - q_2)(\psi_{q_1} + \psi_{q_2} + \psi_{q_1}\psi_{q_2})dx. \tag{3.8}$$

Now $\|\psi_{q_i}\|_{L^2(\Omega)} \le \frac{C}{|\rho_i|}$. Therefore by taking $|l| \to \infty$ we get that

$$\widehat{q_1 - q_2}(k) = 0 \qquad \forall\, k \in \mathbb{R}^n$$

concluding the proof.

We now discuss Theorem 3.3.

Sketch of proof of Theorem 3.3. We outline an alternative proof to the one given by Kohn and Vogelius of Theorem 3.3. In the case $\gamma \in C^\infty(\overline{\Omega})$ we know, by another result of Calderón ([C II]), that Λ_γ is a classical pseudodifferential operator of order 1. Let (x', x^n) be coordinates near a point $x_0 \in \partial\Omega$ so that the boundary is given by $x^n = 0$. If $\lambda_\gamma(x', \xi')$ denotes the full symbol of Λ_γ in these coordinates. It was proved in [S-U III].

$$\lambda_\gamma(x', \xi') = \gamma(x', 0)|\xi'| + a_0(x', \xi') + r(x', \xi') \tag{3.9}$$

where $a_0(x', \xi')$ is homogeneous of degree 0 in ξ' and is determined by the normal derivative of γ at the boundary and tangential derivatives of γ at the boundary. The term $r(x', \xi')$ is a classical symbol of order -1. Then $\gamma\Big|_{\partial\Omega}$ is determined by the principal symbol of Λ_γ and $\frac{\partial\gamma}{\partial x^n}\Big|_{\partial\Omega}$ by the principal symbol and the term homogeneous of degree 0 in the expansion of the full symbol of Λ_γ. More generally the higher order normal derivatives of the conductivity at the boundary can be determined recursively. In [L-U] one can find a more general approach to the calculation of the full symbol of the Dirichlet to Neumann map.

The case $\gamma \in C^1(\overline{\Omega})$ of Lemma (3.3) follows using an approximation argument [S-U III]. For other results and approaches to boundary determination of the conductivity see [Al II], [Br II], [N I].

It is not clear at present what is the optimal regularity on the conductivity for Theorem 3.1 to hold. Chanillo proves in [Ch] that Theorem 3.1 is valid under the assumption that $\Delta\gamma \in F^p$, $p > \frac{n-1}{2}$ where F^p is the Fefferman-Phong class and it is also small in this class. He also presents an argument of Jerison and Kenig that shows that if one assumes $\gamma_i \in W^{2,p}(\Omega)$ with $p > \frac{n}{2}$ then Theorem 3.1 hold. An identifiability result was proven by Isakov [Is I] for conductivities having jump type singularities across a submanifold.

R. Brown [Br I] has shown that theorem 3.1 is valid if one assumes $\gamma_i \in C^{\frac{3}{2}+\epsilon}(\overline{\Omega})$ by using the arguments in [S-U I] combined with Theorem 2.2.

The arguments used in the proofs of Theorems 3.1, 3.2, 3.3 can be pushed further to prove the following stability estimates. For stability estimates for the inverse scattering problem at a fixed energy see [St].

Theorem 3.4 ([Al I]) *Suppose that $s > \frac{n}{2}$ and that γ_1 and γ_2 are C^∞ conductivities on $\overline{\Omega} \subseteq \mathbb{R}^n$ satisfying*

i) $0 < \frac{1}{E} \le \gamma_j \le E$

ii) $\|\gamma_j\|_{H^{s+2}(\Omega)} \le E$

Then there exists $C = C(\Omega, E, n, s)$ and $0 < \sigma < 1$ ($\sigma = \sigma(n,s)$) such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \le C\{|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},\frac{-1}{2}}|^{-\sigma} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2},\frac{-1}{2}}\} \tag{3.10}$$

where $\| \quad \|_{\frac{1}{2},\frac{-1}{2}}$ denotes the operator norm as operators from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{-\frac{1}{2}}(\partial\Omega)$.

This result is a consequence of the next two results.

Theorem 3.5 ([Al I]) *Assume 0 is not a Dirichlet eigenvalue of $\Delta - q_i$, $i = 1, 2$. Let $s > \frac{n}{2}$, $n \ge 3$ and*

$$\|q_j\|_{H^s(\Omega)} \le M.$$

Then there exists $C = C(\Omega, M, n, s)$ and $0 < \sigma < 1$ ($\sigma = \sigma(n,s)$) such that

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \le C(|\log \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\frac{1}{2},\frac{-1}{2}}|^{-\sigma} + \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\frac{1}{2},\frac{-1}{2}}. \tag{3.11}$$

The stability estimate at the boundary is of Hölder type.

Theorem 3.6 ([S-U III]) *Suppose that γ_1 and γ_2 are C^∞ functions on $\overline{\Omega} \subseteq \mathbb{R}^n$ satisfying*

i) $0 < \frac{1}{E} \le \gamma_i \le E$

ii) $\|\gamma_i\|_{C^2(\overline{\Omega})} \leq E$

Given any $0 < \sigma < \frac{1}{n+1}$, *there exists* $C = C(\Omega, E, n, \sigma)$ *such that*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\frac{1}{2}, \frac{-1}{2}} \tag{3.12}$$

and

$$\left\| \frac{\partial \gamma_1}{\partial \nu} - \frac{\partial \gamma_2}{\partial \nu} \right\|_{L^\infty(\partial\Omega)} \leq C \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|^{\sigma}_{\frac{1}{2}, \frac{-1}{2}}. \tag{3.13}$$

The complex geometrical optics solution of Theorems 3.1 and 3.2 were also used by A. Nachman [N I] and R. Novikov [No I] to give a *reconstruction* procedure of the conductivity from Λ_γ.

We first can reconstruct γ at the boundary since $\gamma\big|_{\partial\Omega} |\xi'|$ is the principal symbol of Λ_γ (see (3.9). In other words in coordinates (x', x^n) so that $\partial\Omega$ is locally given by $x^n = 0$ we have

$$\gamma(x', 0) = \lim_{s \to \infty} e^{-is<x', \omega'>} \frac{1}{s} \Lambda_\gamma (e^{is<x', \omega'>})$$

with $\omega' \in \mathbb{R}^{n-1}$ and $|\omega'| = 1$.

In a similar fashion, using (3.9), one can find $\frac{\partial \gamma}{\partial \nu}\Big|_{\partial\Omega}$ by computing the principal symbol of $(\Lambda_\gamma - \gamma\big|_{\partial\Omega} \Lambda_1)$ where Λ_1 denotes the Dirichlet to Neumann map associated to the conductivity 1.

Therefore if we know Λ_γ we can determine Λ_q. We will then show how to reconstruct q from Λ_q. Once this is done, to find $\sqrt{\gamma}$, we solve the problem

$$\begin{aligned} \Delta u - qu &= 0 \text{ in } \Omega, \\ u\Big|_{\partial\Omega} &= \sqrt{\gamma}\Big|_{\partial\Omega}. \end{aligned} \tag{3.14}$$

Let $q_1 = q$, $q_2 = 0$ in formula (3.3). Then we have

$$\int_\Omega quv = \int_{\partial\Omega} (\Lambda_q - \Lambda_0) \left(v\big|_{\partial\Omega}\right) u\big|_{\partial\Omega} dS \tag{3.15}$$

where $u, v \in H^1(\Omega)$ solve $\Delta u - qu = 0$, $\Delta v = 0$ in Ω. Here Λ_0 denotes the Dirichlet to Neumann map associated to the potential $q = 0$. We choose $\rho_i, i = 1, 2$ as in (3.6) and (**??**).

Take $v = e^{x\cdot\rho_1}$, $u := u_\rho = e^{x\cdot\rho_2}(1+\psi_q(x,\rho_2))$ as in Theorem 2.1. By taking $\lim\limits_{|l|\to\infty}$ in (3.15) we conclude

$$\widehat{q}(-k) = \lim_{|l|\to\infty} \int_{\partial\Omega} (\Lambda_q - \Lambda_0)(e^{x\cdot\rho_1}\Big|_{\partial\Omega}) u_\rho\Big|_{\partial\Omega} dS.$$

So the problem is then to recover the boundary values of the solutions u_ρ from Λ_q.

The idea is to find $u_\rho\Big|_{\partial\Omega}$ by looking at the exterior problem. Namely by extending $q = 0$ outside Ω, u_ρ solves

$$\Delta u_\rho = 0 \text{ in } \mathbb{R}^n - \Omega \tag{3.16}$$
$$\frac{\partial u_\rho}{\partial \nu}\Big|_{\partial\Omega} = \Lambda_q(u_\rho\Big|_{\partial\Omega}).$$

Also note that

$$e^{-x\cdot\rho_2} u_\rho - 1 \in L^2_\delta(\mathbb{R}^n). \tag{3.17}$$

Let $\rho \in \mathbb{C}^n - 0$ with $\rho\cdot\rho = 0$. Let $G_\rho(x,y) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ denote the Schwartz kernel of the operator Δ_ρ^{-1}. Then we have that

$$g_\rho(x) = e^{x\cdot\rho} G_\rho(x) \tag{3.18}$$

is a Green's kernel for Δ, namely

$$\Delta g_\rho = \delta_0. \tag{3.19}$$

We shall write the solution of (**??**) and (3.17) in terms of single and double layer potentials using this Green's kernel (also called Faddeev's Green's kernel [F]. For applications to inverse scattering at fixed energy of this Green's kernel see [E-R], [N I], [N-H], [No I,II].)

We define the single and double layer potentials

$$S_\rho f(x) = \int_{\partial\Omega} g_\rho(x-y) f(y) dS_y, \qquad x \in \mathbb{R}^n - \Omega, \tag{3.20}$$

$$D_\rho f(x) = \int_{\partial\Omega} \frac{\partial g_\rho}{\partial \nu}(x-y) f(y) dS_y, \qquad x \in \mathbb{R}^n - \Omega \tag{3.21}$$

$$B_\rho f(x) = p.v. \int_{\partial\Omega} \frac{\partial g_\rho}{\partial \nu}(x-y) f(y) dS_y, \qquad x \in \partial\Omega. \tag{3.22}$$

(3.23)

Nachman showed that $f_\rho = u_\rho\Big|_{\partial\Omega}$ is a solution of the integral equation

$$f_\rho = e^{x\cdot\rho} - (S_\rho \Lambda_q - B_\rho - \frac{1}{2}I) f_\rho. \tag{3.24}$$

Moreover (3.23) is an inhomogeneous integral equation of Fredholm type for f_ρ and it has a unique solution in $H^{\frac{3}{2}}(\partial\Omega)$. The uniqueness of the homogeneous equation follows from the uniqueness of the solutions in Theorem 3.2.

We end this section by considering an extension of Theorem 3.1 to quasilinear conductivities.

Let $\gamma(x,t)$ be a function with domain $\overline{\Omega}\times\mathbb{R}$. Let α be such that $0<\alpha<1$. We assume

$$\gamma \in C^{1,\alpha}(\overline{\Omega}\times[-T,T]), \quad \forall\, T, \tag{3.25}$$

$$\gamma(x,t) > 0 \;\forall\; (x,t)\in\overline{\Omega}\times\mathbb{R}. \tag{3.26}$$

Given $f\in C^{2,\alpha}(\partial\Omega)$, there exists a unique solution of the Dirichlet problem (see [G-T])

$$\begin{cases} \operatorname{div}(\gamma(x,u)\nabla u) &= 0 \quad \text{in } \Omega, \\ u\Big|_{\partial\Omega} &= f. \end{cases} \tag{3.27}$$

Then the Dirichlet to Neumann map is defined by

$$\Lambda_\gamma(f) = \gamma(x,f)\Big|_{\partial\Omega}\frac{\partial u}{\partial\nu}\Big|_{\partial\Omega} \tag{3.28}$$

where u is a solution to (3.27). Sun ([Su I]) proved the following result.

Theorem 3.7 ([Su I]) *Let $n\geq 3$. Assume $\gamma_i\in C^{1,1}(\overline{\Omega}\times[-T,T])\;\forall\,T>0$ and $\Lambda_{\gamma_1}=\Lambda_{\gamma_2}$. Then $\gamma_1(x,t)=\gamma_2(x,t)$ on $\overline{\Omega}\times\mathbb{R}$.*

The main idea is to linearize the Dirichlet to Neumann map at constant boundary data equal to t (then the solution of (3.26) is equal to t). Isakov [Is II] was the first to use a linearization technique to study an inverse parabolic problems associated to non-linear equations. The case of the Dirichlet to Neumann map associated to the Schrödinger equation with a non-linear potential was considered in [I-S], [Is-N] under some assumptions on the potential. We note that, in contrast to the linear case, one cannot reduce the study of the inverse problem of the conductivity equation (3.26) to the Schrödinger equation with a non-linear potential since a reduction similar to (2.1) is not possible in this case. The main technical lemma in the proof of Theorem 3.7 is

Lemma 3.1 *Let $\gamma(x,t)$ be as in (3.25) and (3.26). Let $1 < p < \infty$, $0 < \alpha < 1$. Let us define*

$$\gamma^t(x) = \gamma(x,t). \tag{3.29}$$

Then for any $f \in C^{2,\alpha}(\partial\Omega)$, $t \in \mathbb{R}$

$$\lim_{s\to 0} \|\frac{1}{s}\Lambda_\gamma(t+sf) - \Lambda_{\gamma^t}(f)\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} = 0. \tag{3.30}$$

The proof of Theorem 3.7 follows immediately from the lemma. Namely (3.29) and the hypotheses $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \Lambda_{\gamma_1^t} = \Lambda_{\gamma_2^t}$ for all $t \in \mathbb{R}$. Then using the linear result, Theorem 3.1, we conclude that $\gamma_1^t = \gamma_2^t$ proving the result.

4 The two dimensional case

A. Nachman proved in [N II] that, in the two dimensional case one can uniquely determine conductivities in $W^{2,p}(\Omega)$ for some $p > 1$ from Λ_γ. An essential part of Nachman's argument is the construction of the complex geometrical optics solutions (2.3) for all complex frequencies $\rho \in \mathbb{C}^2 - 0$, $\rho \cdot \rho = 0$ for potentials of the form (2.2). Then he applies the $\overline{\partial}$-method in inverse scattering, pioneered in one dimension by Beals and Coifman [B-C I] and extended to higher dimensions by several authors [B-C II], [A-B-F], [N-A], [N-H], [Ts]. We note one cannot construct these solutions for a general potential for all non-zero complex frequencies as observed by Tsai [Ts].

In fact, the analogous of Theorem 3.2 is open, in two dimensions, for a general potential $q \in L^\infty(\Omega)$. We describe later in part B of this section progress made in the identifiability problem in this case. In part A we outline a different approach to Nachman's result that allows less regular conductivities.

A. The inverse conductivity problem

In this section we describe an extension of Nachman's result to $W^{1,p}(\Omega), p > 2$, conductivities proved by Brown and the author [B-U]. We follow an earlier approach of Beals and Coifman [B-C III] and L. Sung [Sung] who studied scattering for a first order system whose principal part is $\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}$.

Theorem 4.1 *Let $n = 2$. Let $\gamma \in W^{1,p}(\Omega), p > 2$, γ strictly positive. Assume $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then*

$$\gamma_1 = \gamma_2 \text{ in } \overline{\Omega}.$$

We first reduce the conductivity equation to a first order system. We define

$$q = -\frac{1}{2}\partial \log \gamma \tag{4.1}$$

and define a matrix potential Q by

$$Q = \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix}. \tag{4.2}$$

We let D be the operator

$$D = \begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}. \tag{4.3}$$

An easy calculation shows that if u satisfies the conductivity equation $\operatorname{div}(\gamma\nabla u) = 0$, then

$$\begin{pmatrix} v \\ w \end{pmatrix} = \gamma^{\frac{1}{2}} \begin{pmatrix} \partial u \\ \overline{\partial} u \end{pmatrix} \tag{4.4}$$

solves the system

$$D\begin{pmatrix} v \\ w \end{pmatrix} - Q\begin{pmatrix} v \\ w \end{pmatrix} = 0. \tag{4.5}$$

In [B-U] are constructed matrix solutions of (4.5) of the form

$$\psi = m(z,k)\begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\overline{z}k} \end{pmatrix}. \tag{4.6}$$

where $z = x_1 + ix_2$, $k \in \mathbb{C}$ with $m \to 1$ as $|z| \to \infty$ in a sense to be described below. To construct m we solve the integral equation

$$m - D_k^{-1}Qm = 1 \tag{4.7}$$

where, for a matrix-valued function A,

$$D_k A = E_k^{-1} D E_k A \tag{4.8}$$

with

$$E_k A = A^d + \Lambda_k^{-1} A^{\text{off}} \tag{4.9}$$

and

$$\Lambda_k(z) = \begin{pmatrix} e^{i(z\overline{k}+\overline{z}k)} & 0 \\ 0 & e^{-i(zk+\overline{z}\overline{k})} \end{pmatrix}. \tag{4.10}$$

Here A^d denotes the diagonal part of A and A^{off} the antidiagonal part.

Let

$$J = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{4.11}$$

Then we have

$$JA = [J, A] = 2JA^{\text{off}} = -2A^{\text{off}}J. \tag{4.12}$$

where [,] denotes the commutator.

To end with the preliminary notation, we recall the definition of the weighted L^p space

$$L^p_\alpha(\mathbb{R}^2) = \{f; \int (1+|x|^2)^\alpha |f(x)|^p dx < \infty\}.$$

The next result gives the solvability of (4.7) in an appropriate space.

Theorem 4.2 *Let $Q \in L^p(\mathbb{R}^2), p > 2$, and compactly supported. Assume that Q is a hermitian matrix. Choose r so that $\frac{1}{p}+\frac{1}{r} > \frac{1}{2}$ and then β so that $\beta r > 2$. Then the operator $(I - D_k^{-1}Q)$ is invertible in $L^r_{-\beta}$. Moreover the inverse is differentiable in k in the strong operator topology.*

Theorem 4.2 implies the existence of solutions of the form (4.6) with $m - 1 \in L^r_{-\beta}(\mathbb{R}^2)$ with β, r as in Theorem 4.2.
Next we compute $\frac{\partial}{\partial \overline{k}} m(z,k)$.

Theorem 4.3 *Let m be the solution of (4.7) with $m-1 \in L^r_{-\beta}(\mathbb{R}^2)$. Then*

$$\frac{\partial}{\partial \overline{k}} m(z,k) = m(z,\overline{k})\Lambda_k(z)S_Q(k) = 0 \tag{4.13}$$

where the scattering data S_Q is given by (see [B-C III])

$$S_Q(k) = -\frac{1}{\pi} J \int_{\mathbb{R}^2} E_k Q m d\mu \tag{4.14}$$

where $d\mu$ denotes Lebesgue measure in $\mathbb{R}^2$.

Finally, we need an estimate for the growth of m in the variable k. The following result is a straightforward generalization of Proposition 2.23 in [Sung]. (We remark that the proof in [Sung] is incorrect. A corrected proof, kindly provided by L. Sung, appears in [B-U]).

Theorem 4.4 *Let $Q \in L^p(\mathbb{R}^2), p > 2$, and compactly supported. Then there exists $R = R(Q)$ so that for all $q > \frac{2p}{p-2}$,*

$$sup_z \|m(z,\cdot) - 1\|_{L^q\{k;|k|>R\}} \leq C\|Q\|^2_{L^p}$$

where the constants depend on p, q and the diameter of the support of Q.

Outline of proof of Theorem 4.1 We recall (see Theorem 3.3) that if $\gamma_i \in W^{1,p}(\Omega)$ and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\partial^\alpha \gamma_1\Big|_{\partial\Omega} = \partial^\alpha \gamma_2\Big|_{\partial\Omega}$ $\forall\ |\alpha| \leq 1$. Therefore we can extend $\gamma_i \in W^{1,p}(\mathbb{R}^2)$, $\gamma_1 = \gamma_2$ in $\mathbb{R}^2 - \Omega$ and $\gamma_i = 1$ outside a large ball. Thus $Q \in W^{1,p}(\mathbb{R}^2)$. Then Theorems 4.2, 4.3 apply. Now we follow the following steps.

Step 1: $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow S_{Q_1} = S_{Q_2} := S$.

This just follows using that

$$\overline{\partial} m_i - Q_i m_i = 1, i = 1, 2$$

and integrating by parts in (4.14) (in a ball containing the support of Q_i).

Step 2 Using the $\overline{\partial}$-equation (4.13) and Step 1 we conclude that

$$\frac{\partial}{\partial \overline{k}}(m_1 - m_2) - (m_1 - m_2)\Lambda_k(z) S_Q(k) = 0 \tag{4.15}$$

Therefore $m_1 - m_2 \in L^{\widetilde{p}}_{-\beta}(\mathbb{R}^2)$ satisfies the pseudoanalytic equation

$$\overline{\partial}_k(m_1 - m_2) = r(z,k)(m_1 - m_2). \tag{4.16}$$

where $r(z,k) = S_Q(k)\Lambda_k(z)$.

Step 3 We define

$$(\widetilde{m}_1 - \widetilde{m}_2) = (m_1 - m_2)e^{\overline{\partial}^{-1} r}. \tag{4.17}$$

It is easy to check that

$$\overline{\partial}(\widetilde{m}_1 - \widetilde{m}_2) = 0.$$

Then we can conclude that $\widetilde{m}_1 = \widetilde{m}_2$ and therefore $m_1 = m_2$, which in turn easily implies that $Q_1 = Q_2$ and therefore $\gamma_1 = \gamma_2$ by using the following result, combined with Theorems 4.3, 4.4.

Lemma 4.1 *Let $f \in L^2(\mathbb{R}^2)$ and $w \in L^p(\mathbb{R}^2)$ for some finite p. Assume that $we^{\overline{\partial}^{-1} f}$ is analytic. Then $w = 0$.*

The idea of the proof of Lemma 4.1 is the observation that since $r \in L^2(\mathbb{R}^2)$, $u = \overline{\partial}^{-1} r$ is in $VMO(\mathbb{R}^2)$ (the space of functions with vanishing mean oscillation) and thus is $0(\log|z|)$ as $|z| \to \infty$. Hence $e^u w \in L^{\widetilde{p}}$ for $\widetilde{p} > p$. By Liouville's theorem it follows that $e^u w = 0$. The details can be found in [B-U].

We remark that Theorem 3.7 is also valid in the two dimensional case [Su II].

B. The potential case

As we mentioned at the beginning of this section the analog of Theorem 3.2 is unknown at present for a general potential $q \in L^\infty(\Omega)$. By Nachman's result it is true for potentials of the form $q = \frac{\Delta u}{u}$ with $u \in W^{2,p}(\Omega), u > 0$ for some p, $p > 1$. Sun and Uhlmann proved generic uniqueness for pairs of potentials in [Su-U I]. In [Su- U III] it is shown that one can determine the singularities of an L^∞ potentials from the Dirichlet to Neumann map. Namely we have

Theorem 4.5 *Let $\Omega \subset \mathbb{R}^2$, be a bounded open set with smooth boundary. Let $q_i \in L^\infty(\Omega)$ satisfying*

$$\mathcal{C}_{q_1} = \mathcal{C}_{q_2}.$$

Then

$$q_1 - q_2 \in C^\alpha(\Omega)$$

for all $0 \leq \alpha < 1$.

We shall outline in the remaining of this section the proof of an identifiability result near the 0 potential. This proof exhibits some of the features of the proof of Theorem 4.5. We also use directly Calderón's result that the product of harmonic functions are dense in $L^2(\Omega)$.

Theorem 4.6 *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with smooth boundary. Let $q_i \in W^{1,\infty}(\Omega)$, $i = 1, 2$. Then $\exists\ \epsilon(\Omega) > 0$ such that $\|q_i\|_{W^{1,\infty}(\Omega)} < \epsilon(\Omega)$ and*

$$\Lambda_{q_1} = \Lambda_{q_2}.$$

Then

$$q_1 = q_2.$$

We first state an extension of Theorem 3.2 to potentials in some weighted L^p spaces. We also find an asymptotic expansion for the remainder term ψ.

Theorem 4.7 *Let $1 < p < \infty$ and*

$$-\frac{2}{p} < \delta < -1 + \frac{2}{p'} \text{ with } \frac{1}{p} + \frac{1}{p'} = 1. \tag{4.18}$$

There exists a constant $\epsilon(\delta, p)$ such that for every $q \in L^p_{\delta+1}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and for every $k \in \mathbb{C}$ satisfying

$$\frac{\|(1+|x|^2)^{\frac{1}{2}} q\|_{L^\infty(\mathbb{R}^2)}}{|k|} < \epsilon \tag{4.19}$$

then there exists a unique solution to

$$(\Delta - q)u = 0 \text{ in } \mathbb{R}^2 \tag{4.20}$$

such that

$$u = e^{izk}(1 + \psi(z,k)) \tag{4.21}$$

with $\psi \in L^p_\delta(\mathbb{R}^2)$ satisfying

$$\|\psi\|_{L^p_\delta(\mathbb{R}^2)} + \frac{1}{|k|}\|\nabla\psi\|_{L^p_\delta(\mathbb{R}^2)} \frac{\leq C}{|k|}\|q\|_{L^p_{\delta+1}(\mathbb{R}^2)}. \tag{4.22}$$

Furthermore

$$\psi(x,k) = \frac{1}{4}\frac{(\overline{\partial}^{-1} q)}{ik} + b(x,k) \tag{4.23}$$

with

$$\|b\|_{L^p_\delta(\mathbb{R}^2)} + \frac{1}{|k|}\|\nabla b\|_{L^p_{\delta+1}(\mathbb{R}^2)} \leq \frac{C}{|k|^2}\|q\|_{L^p_{\delta+1}(\mathbb{R}^2)} \tag{4.24}$$

for some $C > 0$.

Outline of Proof of Theorem 4.7

We note that in two dimensions for $z, k \in \mathbb{C} - 0$

$$e^{-izk}\Delta e^{izk} f = 4\overline{\partial}(\partial + ik)f. \tag{4.25}$$

We first compute $(\overline{\partial}(\partial+ik))^{-1}$. We observe that

$$\partial(e^{i(kz+\overline{k}\overline{z})}f) = e^{i(kz+\overline{k}\overline{z})}(\partial+ik)f \tag{4.26}$$

where $f \in C_0^\infty(\mathbb{R}^2)$.

We then define

$$(\partial+ik)^{-1}f = e^{-i(kz+\overline{k}\overline{z})}\partial^{-1}(e^{i(kz+\overline{k}\overline{z})}f) \tag{4.27}$$

where $f \in C_0^\infty(\mathbb{R}^2)$.

Because of Lemma 2.3 we have that $(\partial+ik)^{-1}$ extends as a bounded operator from $L^p_{\delta+1}(\mathbb{R}^2)$ and $L^p_\delta(\mathbb{R}^2)$ and

$$\|(\partial+ik)^{-1}\|_{L^p_{\delta+1},L^p_\delta} \leq C_1 \tag{4.28}$$

with C_1 independent of k. Here $\| \ \|_{L^p_{\delta+1},L^p_\delta}$ denotes the operator norm. Also using (4.27) we have that

$$\|D_z(\partial+ik)^{-1}\|_{L^p_{\delta+1},L^p_{\delta+1}} \leq C_2|k| \tag{4.29}$$

where D_z denotes differentiation in any direction and C_2 is independent of k.

The following identities are easy to check.

Let $f \in L^p_{\delta+1}(\mathbb{R}^2)$

$$(\partial+ik)^{-1}f = \frac{1}{ik}(I-\partial(\partial+ik)^{-1})f \tag{4.30}$$

and

$$(\partial+ik)^{-1}f = \frac{f}{ik} - \frac{\partial}{(ik)^2}(I-\partial(\partial+ik)^{-1}))f. \tag{4.31}$$

Therefore it follows from (4.30) that

$$\left(\overline{\partial}(\partial+ik)\right)^{-1}f = \frac{1}{ik}[\overline{\partial}^{-1} - \overline{\partial}^{-1}\partial(\partial+ik)^{-1}]f. \tag{4.32}$$

Using now (4.28) and (4.32) we get

$$\|(\overline{\partial}(\partial+ik))^{-1}\|_{L^p_{\delta+1},L^p_\delta} \leq \frac{C_3}{|k|} \tag{4.33}$$

and

$$\|D_z(\overline{\partial}(\partial + ik))^{-1}\|_{L^p_{\delta+1}, L^p_{\delta+1}} \leq C_4 \tag{4.34}$$

with C_3, C_4 independent of k.

Now substituting (4.21) into (4.20) we get that ψ must satisfy

$$4\overline{\partial}(\partial + ik)\psi = q(\psi + 1) \tag{4.35}$$

or

$$\psi = \frac{1}{4}((\overline{\partial}(\partial + ik))^{-1})(q(1 + \psi)). \tag{4.36}$$

The existence of a unique ψ in $L^p_\delta(\mathbb{R}^2)$ follows easily from a contraction argument using (4.22) and (4.19). Also the estimate (4.22) follows from (4.33), (4.34). To obtain (4.23) and (4.24) we note that (4.31) implies that

$$(\overline{\partial}(\partial + ik))^{-1} f = \frac{\overline{\partial}^{-1} f}{ik} - \frac{\overline{\partial}^{-1}}{(ik)^2}\partial(I - \partial(\partial + ik)^{-1} f). \tag{4.37}$$

According to (4.36)

$$\psi = \frac{1}{4}((\overline{\partial}(\partial + ik)^{-1} q + (\overline{\partial}(\partial + ik))^{-1}(q\psi)). \tag{4.38}$$

The first term in (4.38) is

$$\frac{1}{4}(\overline{\partial}(\partial + ik)^{-1})q = \frac{1}{4}\frac{\overline{\partial}^{-1} q}{ik} - \frac{\overline{\partial}^{-1}\partial(I - \partial(\partial + ik)^{-1})q}{4ik^2}. \tag{4.39}$$

and the second term in (4.38) satisfies the estimate (4.24) because of (4.33), (4.37), (4.22), concluding the proof.

Outline of proof of Theorem 4.6

The proof follows using a "compactness" lemma for elements orthogonal to the product of solutions of the Schrödinger equation. Specifically

Lemma 4.2 *Let* $\Omega \subseteq \mathbb{R}^2$ *be a bounded domain with smooth boundary. Let* $0 < s < 1$. *Let* q_i, $i = 1, 2$ *satisfy*

$$\|q_i\|_{L^4(\Omega)} \leq M. \tag{4.40}$$

Then there exists a constant $C = C(\Omega, s, M)$ *such that if*

$$\int_\Omega f u_1 u_2 = 0 \tag{4.41}$$

for all $u_i \in H^1(\Omega)$ *solution of* $\Delta u_i - q_i u_i = 0, i = 1, 2$ *with* $f \in L^2(\Omega)$*, then* $f \in H^s(\Omega)$ *and*

$$\|f\|_{H^s(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{4.42}$$

We now use the above lemma to conclude the proof of Theorem 4.6. Suppose that Theorem 4.5 is false. Then $\exists$ a sequence of pairs $\{q_1^{(n)}, q_2^{(n)}\}$ with $q_1^{(n)} \neq q_2^{(n)}$ for all n approaching zero in $W^{1,\infty}(\Omega)$ and satisfying

$$\Lambda_{q_1^{(n)}} = \Lambda_{q_2^{(n)}}. \tag{4.43}$$

(The set of q's for which $\mathcal{C}_q = \Lambda_q$ is dense in $W^{1,\infty}(\Omega)$). Now we have in this case

$$\int_\Omega (q_1^{(n)} - q_2^{(n)}) u_1^{(n)} u_2^{(n)} = 0 \tag{4.44}$$

for all $u_i^{(n)}$ solution of

$$(\Delta - q_i^{(n)}) u_i^{(n)} = 0 \text{ in } \Omega, \quad i = 1, 2. \tag{4.45}$$

Let

$$f_n = \frac{q_1^{(n)} - q_2^{(n)}}{\|q_1^{(n)} - q_2^{(n)}\|_{L^2(\Omega)}}. \tag{4.46}$$

It follows from Rellich's Lemma and (4.42) that the f_n have a convergent subsequence $f_{n(i)} \to f \in L^2(\Omega)$ with $\|f\|_{L^2(\Omega)} = 1$. Now, let u and v be arbitrary harmonic functions in Ω with boundary values of h and g. Let $u_1^{(n)}$ and $v_2^{(n)}$ denote the solutions to (4.45) with the same boundary values as u and v, respectively, then

$$\begin{aligned} \Delta(u - u_1^{(n)}) &= q_1^{(n)}(u - u_1^{(n)}) - q_1^{(n)} u \\ (u - u_1^{(n)})\Big|_{\partial\Omega} &= 0. \end{aligned}$$

Hence

$$\begin{aligned} \|u - u_1^{(n)}\|_{W^{2,2}(\Omega)} &\leq C_1(\|q_1^{(n)}(u - u_1^{(n)})\|_{L^2(\Omega)} + \|q_1^{(n)}\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}) \\ &\leq C_1(\|q_1^{(n)}\|_{L^\infty(\Omega)}\|(u - u_1^{(n)})\|_{L^2(\Omega)} + \|q_1^{(n)}\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}). \end{aligned} \tag{4.47}$$

For $\|q_1^{(n)}\|_{L^\infty(\Omega)}$ small enough

$$\|u - u_1^{(n)}\|_{W^{2,2}(\Omega)} \leq C_2 \|q_1^{(n)}\| \, \|u\|_{L^2(\Omega)}$$

so that

$$u_1^{(n)} \xrightarrow{W^{2,2}(\Omega)} u$$

and similarly

$$v_2^{(n)} \xrightarrow{W^{2,2}(\Omega)} v.$$

In particular, since convergence in $W^{2,2}(\Omega)$ implies convergence in $L^4(\Omega)$ ([G-T])

$$u_1^{(n)} v_2^{(n)} \xrightarrow{L^2(\Omega)} uv. \tag{4.48}$$

We know that

$$0 = \int_\Omega f_n u_1^{(n)} v_2^{(n)}$$

so, it follows from (4.44) and (4.48) that

$$0 = \int_\Omega fuv. \tag{4.49}$$

For all harmonic u and v. By Calderón's argument we get $f = 0$, a contradiction.

Proof of Lemma 4.2 We begin by choosing u_1 and u_2 to be solutions to

$$\Delta u_i - q_i u_i = 0, \quad i = 1, 2$$

in $\mathbb{R}^2$ of the form (4.21), with q_i extended to be zero outside Ω, and δ satisfying (4.18) with $p = 4$ of the form

$$\begin{aligned} u_1 &= e^{izk}(1 + \psi_1(x,k)) \\ u_2 &= e^{i\bar{z}\bar{k}}(1 + \psi_2(x,k) \end{aligned}$$

and, in order to satisfy (4.19),

$$|k| \geq \frac{\|(1+|x|^2)^{1/2} q\|_{L^\infty(\Omega)}}{\epsilon} =: R. \tag{4.50}$$

Hence, using the expansion (4.23) with $g_i = \overline{\partial}^{-1} q_i$, $w_i := \psi_{q_i}$ (4.41) becomes

$$
(4.51) \qquad \begin{array}{rcl} -\int_{\mathbb{R}^2} e^{i(zk+\overline{z}\overline{k})} f & = & \int_{\mathbb{R}^2} f\psi_1\psi_2 e^{(izk+\overline{z}\overline{k})} + \int_{\mathbb{R}^2} f \frac{(g_1+g_2)}{4ik} e^{i(zk+\overline{z}\overline{k})} \\ & + & \int_{\mathbb{R}^2} f(b_1+b_2) e^{i(zk+\overline{z}\overline{k})} \end{array}
$$

where f is extended to be zero outside Ω. We denote by I_1, I_2, and I_3 the three terms on the right hand side of (4.51). We have

$$
\begin{array}{rcl} |I_1| & \leq & C_1 \|f\|_{L^2(\Omega)} \|\psi_1\psi_2\|_{L^2(\Omega)} \\ & \leq & C_1 \|f\|_{L^2(\Omega)} \|\psi_1\|_{L^4(\Omega)} \|\psi_2\|_{L^4(\Omega)} \end{array}
$$

which implies, in view of (4.22),

$$
|I_1| \leq \frac{C_2(\Omega)}{|k|^2} \|f\|_{L^2(\Omega)} \|q_1\|_{L^4_{\delta+1}((\mathbb{R}^2)} \|q_2\|_{L^4_{\delta+1}(\mathbb{R}^2)}
$$

so that, for $0 < s < 1$

$$
(4.52) \qquad \|\,|k|^s I_1\|_{L^2(|k|\geq R)} \leq C_3(\Omega, s) \|q_1\|_{L^4_{\delta+1}(\mathbb{R}^2)} \|q_2\|_{L^4_{\delta+1}(\mathbb{R}^2)} \|f\|_{L^2(\Omega)}.
$$

In addition,

$$
I_2 = \frac{[f(g_1+g_2)]^\wedge(k)}{4ik} \text{ with } \widehat{f}(k) := \int_{\mathbb{R}^2} e^{i(zk+\overline{z}\overline{k})} f
$$

so that

$$
(4.53) \qquad \begin{array}{rcl} \|\,|k|^s I_2\|_{L^2(|k|\geq R)} & \leq & \|[f(g_1+g_2)]^\wedge\|_{L^2(|k|\geq R)} \\ & \leq & \|f(g_1+g_2)\|_{L^2(\mathbb{R}^2)} \\ & \leq & \|f\|_{L^2(\Omega)} \|g_1+g_2\|_{L^\infty(\Omega)} \\ & \leq & C_4 \|f\|_{L^2(\Omega)} \|q_1+q_2\|_{L^4_\delta(\Omega)}. \end{array}
$$

In summary,

$$
(4.54) \qquad \|\,|k|^s I_2\|_{L^2(|k|\geq R)} \leq C_6 \|f\|_{L^2(\Omega)} (\|q_1\|_{L^4_{\delta+1}(\mathbb{R}^2)} + \|q_2\|_{L^4_{\delta+1}(\mathbb{R}^2)})
$$

To estimate I_3, we use (4.24). We obtain

$$
(4.55) \qquad \begin{array}{rcl} |k|^s |I_3| & \leq & |k|^s \|f\|_{L^2(\Omega)} (\|b_1\|_{L^2_\delta(\mathbb{R}^2)} + \|b_2\|_{L^2_\delta(\mathbb{R}^2)}) \\ & \leq & |k|^{s-2} \|f\|_{L^2(\Omega)} (\|q_1\|_{L^2_{\delta+1}(\mathbb{R}^2)} + \|q_2\|_{L^2_{\delta+1}(\mathbb{R}^2)}). \end{array}
$$

Finally, we note that

$$
(4.56) \qquad \|f\|^2_{H^s} \leq 2(\|\widehat{f}\|^2_{L^2(|k|\leq R)} (1+R^2)^s + \|\,|k|^s \widehat{f}\|_{L^2(|k|\geq R)}).
$$

Combining (4.56) with (4.51), (4.52), (4.54), and (4.55), where R is chosen in (4.50), gives (4.20).

5 First order perturbations of the Laplacian

In this section we consider inverse boundary value problems associated to first order perturbations of the Laplacian. We consider two important cases arising in applications. We first consider the Schrödinger equation in the presence of a magnetic potential. The problem is to determine both the electric and magnetic potential of a medium by making measurements at the boundary of the medium. The second example involves an elliptic system . We consider an elastic body. The problem is to determine the elastic parameters of this body by making displacements and traction measurements at the boundary. Another important inverse boundary value problem involving the system of Maxwell's equations is the determination of the electric permittivity, magnetic permeability and electrical conductivity of a body by measuring the tangential component of the electric and magnetic field ([S-C-I]). A global identifiability result and a reconstruction method can be found in [O-P-S]. This problem can also be reduced to construct complex geometrical optics solutions for first order perturbations of the Laplacian ([Su-U IV]). Recently in [O-S] it was shown that, by considering a larger system, one can reduce the problem of constructing the complex geometrical optics solutions for Maxwell's equations to a zeroth order perturbation of the Laplacian. Then the solutions can be constructed as in section 2.

5.1 A. The Schrödinger equation with magnetic potential

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with smooth boundary. The Schrödinger equation in a magnetic field is given by

$$H_{\vec{A},q} = \sum_{j=1}^{n} \left(\frac{1}{i} \frac{\partial}{\partial x_j} + A_j(x) \right)^2 + q(x), \quad i = \sqrt{-1}, \tag{5.1}$$

where $\vec{A} = (A_1, A_2, \ldots, A_n) \in C^1(\bar{\Omega})$ is the magnetic potential and $q \in L^\infty(\Omega)$ is the electric potential. The magnetic field is the rotation of the magnetic potential, $\text{rot}(\vec{A})$.

We assume that $\vec{A}$ and q are real-valued function and thus (5.1) is self-adjoint. We also assume that zero is not a Dirichlet eigenvalue of (5.1) on Ω (or as in section 2, one can consider the Cauchy data associated to $H_{\vec{A},q}$), so that the boundary value problem

$$\begin{cases} H_{\vec{A},q} u &= 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= f \quad \in H^{\frac{1}{2}}(\partial\Omega) \end{cases} \tag{5.2}$$

has a unique solution $u \in H^1(\Omega)$. The Dirichlet-to-Neumann map $\Lambda_{\vec{A},q}$ which maps $H^{\frac{1}{2}}(\partial\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$, is defined by

$$\Lambda_{\vec{A},q} : f \to \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} + i(\vec{A}\cdot\nu)f, \quad f \in H^{\frac{1}{2}}(\partial\Omega), \tag{5.3}$$

where u is the unique solution to (5.2), and ν is the unit outer normal on $\partial\Omega$.

The inverse boundary value problem for (5.1) is to recover information of $\vec{A}$ and q from knowledge of $\Lambda_{\vec{A},q}$.

It is easy to see [Su II] that the the Dirichlet to Neumann map $\Lambda_{\vec{A},q}$ is invariant under a gauge transformation in the magnetic potential: $\vec{A} \to \vec{A} + \nabla g$, where $g \in C^1_\Omega$, where we denote

$$C^s_\Omega = \{f \in C^s(\mathbb{R}^n), \mathrm{supp} f \subset \Omega\}. \tag{5.4}$$

In fact if we consider u as in (5.2), then $v = e^{ig}u$ solves $H_{\vec{A}+\nabla g,q} v = 0$ in Ω and $\Lambda_{\vec{A},q} f = \Lambda_{\vec{A}+\nabla g,q} f$. Thus, $\Lambda_{\vec{A},q}$ carries information about the magnetic field instead of information about $\vec{A}$. The natural question is whether $\Lambda_{\vec{A},q}$ determines uniquely $\mathrm{rot}(\vec{A})$ and q. In [Su II], this question was answered affirmatively for $\vec{A}$ in the C^2_Ω class and q in the $L^\infty(\Omega)$ class, under the assumption that $\mathrm{rot}(\vec{A})$ is small in the L^∞ topology.

The smallness assumption in Sun's result was used to construct complex geometrical optics solutions similar to (2.3) in this case. In section 6 we'll describe how Nakamura and Uhlmann [N-U I] constructed these types of solutions without the smallness assumption on $\mathrm{rot}(\vec{A})$. This combined with the methods of [Su II] lead to the following result [N-Su-U].

Theorem 5.1 *Let $\vec{A}_j \in C^\infty(\bar{\Omega})$, $q_j \in C^\infty(\bar{\Omega})$, $j = 1, 2$. Assume that zero is not a Dirichlet eigenvalue of $H_{\vec{A}_j,q_j}$, $j = 1, 2$. If*

$$\Lambda_{\vec{A}_1,q_1} = \Lambda_{\vec{A}_2,q_2},$$

then

$$\mathrm{rot}(\vec{A}_1) = \mathrm{rot}(\vec{A}_2) \text{ and } q_1 = q_2 \text{ in } \Omega.$$

If we assume $\vec{A}_j \in C^\infty_\Omega$, it was proved in [N-Su-U] that Theorem 5.1 holds even for $q_j \in L^\infty(\Omega)$. We have

Theorem 5.2 *Let* $\vec{A}_j \in C^{\infty}_{\Omega}$, $q_j \in L^{\infty}(\Omega)$, $j = 1, 2$. *Assume that zero is not a Dirichlet eigenvalue of* $H_{\vec{A}_j, q_j}$, $j = 1, 2$. *If*

$$\Lambda_{\vec{A}_1, q_1} = \Lambda_{\vec{A}_2, q_2},$$

then

$$\operatorname{rot}(\vec{A}_1) = \operatorname{rot}(\vec{A}_2) \text{ and } q_1 = q_2 \text{ in } \Omega.$$

The inverse scattering problem at a fixed energy in this case has been considered in [E-R].

C. Tolmasky ([To]) reduced the regularity of $\vec{A}_j$ in Theorem 5.2 to just $\vec{A}_j \in C^1_{\Omega}$. He constructs the complex geometrical optics solutions under weaker regularity conditions. We shall outline his approach in the next section.

B. Inverse boundary value problems for elastic materials

We assume now that Ω is an elastic material, that is, a deformed shape will try to come back to its original shape. Let $u(x)$ denote the displacement of the point x under the deformation. The undeformed domain is called the reference configuration space. The *linear strain tensor*

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right), \quad i, j = 1, \ldots, n \tag{5.5}$$

measures the rate of deformation with respect to the Euclidean metric for small deformations. Under the assumption of no-body forces acting on Ω, the equation of equilibrium in the reference configuration is given by the generalized Hooke's law (see [Ci] for an excellent treatment of elasticity theory)

$$L_C u = \operatorname{div} \sigma(u) = 0 \text{ in } \Omega, \tag{5.6}$$

where $\sigma(u)$ is a symmetric two-tensor called *the strain tensor*. The *elastic tensor* C is a fourth order tensor which satisfies

$$\sigma_{ij}(u) = \sum_{k,l=1}^{n} C_{ijkl}(x)\varepsilon_{kl}(u), \qquad i, j = 1, \ldots, n. \tag{5.7}$$

We shall assume that the elastic tensor satisfies the hyperelasticity condition ([Ci], Chapter 4)

$$C_{ijkl}(x) = C_{klij}(x) \quad \forall\, x \in \bar{\Omega}. \tag{5.8}$$

We also assume that C satisfies the strong convexity condition: there exists $\delta > 0$ such that

$$\sum_{i,j,k,l=1}^{n} C_{ijkl}(x) t_{ij} t_{kl} \geq \delta \sum_{i,j=1}^{n} t_{ij}^2, \quad x \in \bar{\Omega} \tag{5.9}$$

for any real-symmetric matrix $(t_{ij})_{1 \leq i,j \leq n}$. Condition (5.9) guarantees the unique solvability of the Dirichlet problem

$$\begin{cases} \operatorname{div} \sigma(u) &= 0 \qquad \text{in } \Omega, \\ u\big|_{\partial\Omega} &= f. \end{cases} \tag{5.10}$$

The Dirichlet integral associated to (5.10) is given by

$$W_C(f) = \sum_{i,j,k,l=1}^{n} \int_\Omega C_{ijkl}(x) \frac{\partial u_k}{\partial x_l} \frac{\partial u_i}{\partial x_j} dx \tag{5.11}$$

with u solution of (5.10). Physically, $W_C(f)$ measures the deformation energy produced by the displacement f at the boundary.

Applying the divergence theorem we have that

$$W_C(f) = \sum_{i=1}^{n} \int_\Omega (\Lambda_C(f))_i f_i \, dx, \tag{5.12}$$

where

$$(\Lambda_C(f))_i = \sum_{j,k,l=1}^{n} \nu^j C_{ijkl} \frac{\partial u_k}{\partial x_l}\Big|_{\partial\Omega}, \qquad i = 1, \ldots, n \tag{5.13}$$

with u solution of (5.10) and ν denotes the unit outer normal to $\partial\Omega$. In other words, Λ_C is the linear operator associated to the quadratic form W_C. The map

$$f \xrightarrow{\Lambda_C} \Lambda_C(f) \tag{5.14}$$

is the *Dirichlet* to *Neumann* map in this case. It sends the displacement at the boundary to the corresponding traction at the boundary. The inverse problem we consider in this subsection is whether we can determine C from Λ_C.

We shall assume that C is isotropic, i.e., satisfies

$$C_{ijkl}(x) = \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{5.15}$$

where δ_{ik} denotes the Krönecker delta.

In this case the strain tensor takes the form

$$\sigma(u) = \lambda(x)(\text{trace } \varepsilon(u)) + 2\mu(x)\text{div } (u). \tag{5.16}$$

The strong-convexity condition (5.9) is equivalent to

$$n\lambda + 2\mu > 0, \quad \mu > 0 \text{ in } \bar{\Omega}. \tag{5.17}$$

The main known results for identifiability of C from Λ_C are

Theorem 5.3 (N-U I) *Let $n \geq 3$, $C_j \in C^\infty(\bar{\Omega})$, isotropic elastic tensor. $j = 1, 2$. Assume*

$$\Lambda_{C_1} = \Lambda_{C_2}.$$

Then $C_1 = C_2$

Theorem 5.4 (N-U III) *Let $n = 2$ and C_j as in Theorem 3.4 with Lamé parameters $\lambda_j, \mu_j, j = 1, 2$. There exists $\epsilon > 0$ such that if*

$$\|(\lambda_j, \mu_j) - (\lambda_0, \mu_0)\|_{W^{31,\infty}(\Omega)} < \epsilon, \quad j = 1, 2$$

and $\Lambda_{C_1} = \Lambda_{C_2}$ then $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$. Here (λ_0, μ_0) denotes a constant and $\|u\|_{W^{31,\infty}(\Omega)} = \sup\limits_{\substack{x\in\Omega \\ |\alpha|\leq 31}} |\partial^\alpha u(x)|$.

The global uniqueness result Theorem 5.3 is the analog to Theorem 3.1 for the inverse conductivity problem. Theorem 5.4 is analogous to the local result, Theorem 4.6. We remark that there is no known global result in two dimensions similar to the one proven by Nachman [N II] for the inverse conductivity problem. Theorem 5.3 has been extended in [N-S] to a class of non-linear elastic materials, the so-called St. Venant-Kirchohoff's materials, using a linearization technique similar to Lemma 3.

We now indicate the main steps in the proofs of Theorems 5.3 and 5.4. The first observation is that under the hypothesis of Theorems 5.3, 5.4 we can prove an identity involving $\lambda_1 - \lambda_2$, $\mu_1 - \mu_2$.

Lemma 5.1 *Let $u^{(i)}$, $i = 1, 2$ be solutions*

$$\text{div } \sigma(u^{(i)}) = 0 \textit{ in } \Omega,$$

with $u^{(i)} \in H^1(\Omega)$, $i = 1, 2$. *Assume* $\Lambda_{C_1} = \Lambda_{C_2}$. *Then*

$$E(u^{(1)}, u^{(2)}) := \int_\Omega (\lambda_1 - \lambda_2) \text{div } u^{(1)} \cdot \overline{\text{div } u^{(2)}} dx + 2 \int_\Omega (\mu_1 - \mu_2) \varepsilon(u^{(1)}) \cdot \overline{\varepsilon(u^{(2)})} dx = 0. \tag{5.18}$$

The proof of the Lemma follows readily by applying the divergence Theorem and the boundary determination result of [N-U II], which is the analog of Theorem 3.3. Namely under the hypothesis of Theorems 5.1, 5.2 we have that the Taylor series of λ_1, μ_1 and λ_2, μ_2 coincide. (It is only needed $\partial^\alpha \mu, \partial^\alpha \lambda$, $|\alpha| \leq 1$).

The problem is now to find "enough" solutions of $L_{C_i} u^{(i)} = 0$ in Ω to conclude that the Lamé parameters coincide in Ω.

It is quite difficult to construct solutions of the form (2.3) directly for the elasticity system. In the paper [N-U I] a reduction to a system with principal part the biharmonic operator is made by multiplying L_C on the left by an explicit second order system T_C to get

$$T_C L_C = \Delta^2 + M_1(x, D)\Delta + M_2(x, D) \tag{5.19}$$

with $M_i(x, D)$ an $n \times n$ system of order i, $i = 1, 2$. Then to construct solutions of $L_C u = 0$ it is enough to construct solutions of $Mu = (\Delta^2 + M_1(x, D)\Delta + M_2(x, D))u = 0$. By introducing a new dependent variable $v = \Delta u$ we want to find solutions of the $2n \times 2n$ system

$$\Delta \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -I \\ 0 & M_2 \Delta^{-1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.20}$$

where Δ^{-1} denotes the inverse of the Laplacian. Notice that (5.20) is a first order system perturbation of the Δ with the $0^{\underline{\text{th}}}$ order perturbation being a pseudodifferential operator.

Ikehata [I] reduced the elasticity system to an $(n+1) \times (n+1)$ differential system as follows.

Lemma 5.2 ([I]) *Let* $\begin{pmatrix} u \\ f \end{pmatrix}$ *with* $u = (u_1, \dots, u_n)^T$ *be a solution of the* $(n+1) \times (n+1)$ *system*

$$\Delta I_{n+1} + V^1(x) \begin{pmatrix} \nabla f \\ \nabla \cdot u \end{pmatrix} + V^0(x) \begin{pmatrix} u \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.21}$$

with

$$V^1(x) = \begin{pmatrix} 2\mu^{-\frac{1}{2}}(-\nabla^2 + I_n \Delta)\mu^{-1} & -\nabla \log \mu \\ 0 & \frac{\lambda+\mu}{\lambda+2\mu}\mu^{\frac{1}{2}} \end{pmatrix} \tag{5.22}$$

$$V^0(x) = \begin{pmatrix} -\mu^{-\frac{1}{2}}(2\nabla^2 + I_n\Delta)\mu^{\frac{1}{2}} & 2\mu^{-\frac{5}{2}}(\nabla^2 - \Delta\ I_n)\mu\nabla\mu \\ -\frac{\lambda-\mu}{\lambda+2\mu}(\nabla\mu^{\frac{1}{2}})^t & -\mu\Delta\mu^{-1} \end{pmatrix} \tag{5.23}$$

and I_k denotes the identity $k \times k$ matrix. Then

$$w = \mu^{-\frac{1}{2}}u + \mu^{-2}\nabla(\mu f) \tag{5.24}$$

satisfies

$$L_C w = 0. \tag{5.25}$$

Therefore we are reduced to finding "enough" solutions of the first order system (5.21) which we rewrite as

$$(\Delta + P^{(1)}(x, D))v = 0 \tag{5.26}$$

with $P^{(1)}(x, D)$ a first order $(n+1) \times (n+1)$ differential system.

We shall indicate in section 6, how to construct these complex geometrical optics solutions.

6 Complex geometrical optics solutions for first order perturbations of the Laplacian

In this section we outline the construction of complex geometrical optics solutions for first order perturbations of the Laplacian. For simplicity we do this for scalar equations. A similar method applies to the first order system (5.21) arising from the elasticity system([N-U I]) Let us consider an operator of the form

$$P(x, D) = \Delta I + P^{(1)}(x, D_x) \tag{6.1}$$

where $P^{(1)}(x, D_x)$ is a first order scalar system with smooth coefficients in $\mathbb{R}^n$ and I denotes the identity matrix. Let $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0, \rho \neq 0$.

In this section, for $|\rho|$ sufficiently large we shall outline the method of [N-U] to construct solutions of

$$P(x, D)u = 0 \tag{6.2}$$

in compact sets of $\mathbb{R}^n$ of the form

$$u = e^{x \cdot \rho}v(x, \rho) \tag{6.3}$$

with a fairly precise control of the behavior of $v(x, \rho)$ as $|\rho| \to \infty$.

We will construct $v(x, \rho)$ as a solution of

$$P_\rho(x, D)v = 0 \tag{6.4}$$

where

$$P_\rho(x, D) = \Delta_\rho + P_\rho^{(1)}(x, D) \tag{6.5}$$

and

$$\Delta_\rho = e^{-x\cdot\rho}\Delta(e^{x\cdot\rho}), P_\rho^{(1)}(x, D) = e^{-x\cdot\rho}P^{(1)}(x, D)(e^{x\cdot\rho}).$$

As it was shown in Section 2 we have precise estimates for Δ_ρ^{-1}. The problem is that derivatives of $\Delta_\rho^{-1}f$ don't decay for large ρ. Also $P_\rho^{(1)}$ involves terms growing in ρ. The goal is to, somehow, get rid of the first order terms in (6.5). Roughly speaking, we will construct invertible operators A_ρ, B_ρ, and an operator C_ρ of "lower order" so that

$$P_\rho A_\rho = B_\rho(\Delta_\rho + C_\rho) \tag{6.6}$$

We will then construct solutions of $P_\rho v_\rho = 0$ of the form

$$v_\rho = A_\rho w_\rho$$

with w_ρ solution of $(\Delta_\rho + C_\rho)w_\rho = 0$.

We'll accomplish (6.6) using the theory of pseudodifferential operators depending on the complex vector ρ. The main point is to regard the variables ξ and ρ in equal footing. We digress to discuss the main features of this theory. For more details see for instance [Sh]. Let

$$Z = \{\rho \in \mathbb{C}^n; |\rho| \geq 1, \rho \cdot \rho = 0\}$$

Definition 6.1 *Let $l \in \mathbb{R}$, $0 \leq \delta < 1$, $U \subset \mathbb{R}^m$, U open. We say that $a_\rho \in S_\delta^l(U, Z) \Leftrightarrow$ $\forall \rho \in Z$ fixed, $a_\rho(x, \xi) \in C^\infty(U \times \mathbb{R}^n)$; $\forall\ \alpha \in \mathbb{Z}_+^n$, $\forall\ \beta \in \mathbb{Z}_+^m$, $\forall\ K \subset U$, K compact $\exists C_{\alpha,\beta,K} > 0$ such that*

$$\sup_{x\in K} |\partial_\xi^\alpha \partial_x^\beta a_\rho(x, \xi)| \leq C_{\alpha,\beta,k}(1 + |\xi| + |\rho|)^{l+\beta\delta-|\alpha|} \qquad \forall \rho \in Z, \xi \in \mathbb{R}^n.$$

EXAMPLE: We have that

$$\widetilde{r}_\rho(x, \xi) = -|\xi|^2 + 2i\xi \cdot \rho \in S_0^2(\mathbb{R}^n \times Z)$$

since it is homogeneous of degree 2 in (ξ, ρ). Notice, however, that $\widetilde{r}_\rho$ is not elliptic. In fact if $n = 3$ and we take $\operatorname{Re}\rho = s(1, 0, 0)$, $\operatorname{Im}\rho = s(0, 1, 0)$, $s \in \mathbb{R}$, then the zeros of $\widetilde{r}_\rho(\xi)$ is a codimension two circle in the plane $\xi_1 = 0$ centered at the point $(0, -s, 0)$ of radius s.

Definition 6.2 *Let $U \subseteq \mathbb{R}^n$, U open, $a_\rho \in S^l_\delta(U,Z)$. We define the operator $A_\rho \in L^l_\delta(U,Z)$ by*

$$A_\rho f(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi} a_\rho(x,\xi)\widehat{f}(\xi)d\xi, \quad f \in C_0^\infty(U). \tag{6.7}$$

The kernel of A_ρ is given by

$$k_{A_\rho}(x,y) = \frac{1}{(2\pi)^n}\int e^{i(x-y)\cdot\xi} a_\rho(x,\xi)d\xi \tag{6.8}$$

where the integral in (6.8) is interpreted as an oscillatory integral. A_ρ extends continuously as a linear operator

$$A_\rho : \mathcal{E}'(\mathcal{U}) \to \mathcal{D}'(\mathcal{U})$$

where $\mathcal{E}'(\mathcal{U})$ (resp. $D'(U)$) denotes the space of compactly supported distributions (resp. distributions).

As usual, it is easy to check that if $a_\rho \in S^l_\delta(U,Z) \quad \forall l$, then $A_\rho : \mathcal{E}'(\mathcal{U}) \to \mathcal{C}^\infty(\mathcal{U})$, i.e. A_ρ is a smoothing operator.

Definition 6.3 *We say that A_ρ is uniformly properly supported if* supp k_{A_ρ} *is contained in a fixed neighborhood V of the diagonal in $U \times U$ for all $\rho \in Z$, so that $\forall K \subset U$, K compact, V intersected with $\Pi^{-1}(K)$ is compact where Π denotes either one the projections of $U \times U$ onto U.*

Proposition 6.1 *Let $A_\rho \in L^m_\delta(U,Z)$. Then we can write*

$$A_\rho = B_\rho + R_\rho$$

with $B_\rho \in L^m_\delta$ uniformly properly supported and R_ρ smoothing.

We shall assume from now on that all pseudodifferential operators are uniformly properly supported.

Definition 6.4 *Let $A_\rho \in L^m_\delta(U,Z)$ as in (6.7).*

a) Then the full symbol of A_ρ is given by $\widetilde{\sigma}_m(A_\rho)(x,\xi) = a_\rho(x,\xi)$.

b) The principal symbol of A_ρ is given by

$$\sigma_m(A_\rho)(x,\xi) = a_\rho(x,\xi) \ mod \ S^{m-1}_\delta(U,Z).$$

The functional calculus for pseudodifferential operators depending on a parameter is completely analogous to the standard calculus. Namely we have

Theorem 6.1 *Let $A_\rho \in L^m_\delta(U,Z)$, $B_\rho \in L^{\widetilde{m}}_\delta(U,Z)$. Then*

a) $A_\rho B_\rho \in L^{m+\widetilde{m}}_\delta(U,Z)$

b) $\widetilde{\sigma}_{m+\widetilde{m}}(A_\rho B_\rho) \sim \Sigma \frac{1}{\alpha!} D^\alpha_\xi \widetilde{\sigma}_m(A_\rho)\partial^\alpha_x \widetilde{\sigma}_{\widetilde{m}}(B_\rho)$

c) $\sigma_{m+\widetilde{m}}(A_\rho B_\rho) = \sigma_m(A_\rho)\sigma_{\widetilde{m}}(B_\rho)$

d) $\sigma_{m+\widetilde{m}}([A_\rho, B_\rho]) = H_{\sigma_m(A_\rho)}\sigma_{\widetilde{m}}(B_\rho)$ *where* $[A_\rho, B_\rho]$ *denotes the commutator and* H_p *denotes the Hamiltonian vector associated to* p*, i.e.*

$$H_p = \sum_{j=1}^{n}\left(\frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}\right).$$

Finally we shall use the following continuity property of A_ρ's on Sobolev spaces (see [Sh]).

Theorem 6.2 *Suppose $l \le 0$, K a compact subset of $\mathbb{R}^{2n}$ and $A_\rho \in L^l_\delta(\mathbb{R}^n, Z)$ with $\operatorname{supp} K_{A_\rho} \subset K$, $\quad \forall \rho \in Z \Rightarrow \forall\, k \in \mathbb{R}$, A_ρ is a bounded operator from $H^k(\mathbb{R}^n)$ to $H^k(\mathbb{R}^n)$ and $\exists C_{k,K} \ge 0$ such that*

$$\|A_\rho\|_{k,k} \le C_{k,K}|\rho|^l \;\forall\; \rho \in Z$$

where $\|A_\rho\|_{k,k}$ denotes the operator norm.

We define

$$\Lambda^s_\rho \in L^s_0(\mathbb{R}^n, Z) \tag{6.9}$$

by

$$\sigma(\Lambda^s_\rho) = (|\xi|^2 + |\rho|^2)^{\frac{s}{2}}. \tag{6.10}$$

We use this to get a first order equation. Let $\widetilde{P}_\rho = P_\rho \Lambda^{-1}_\rho = \widetilde{\Delta}_\rho + \widetilde{P}^{(1)}_\rho(x,D)$ with $\widetilde{\Delta}_\rho = \Delta_\rho \Lambda^{-1}_\rho$, $\widetilde{P}^{(1)}_\rho = P^{(1)}_\rho \Lambda^{-1}_\rho$. A key ingredient in the construction of complex geometrical optics solutions is the following result proven in [N-U I].

Theorem 6.3 (Intertwining property) $\forall$ *positive integer* N $\exists A_\rho, B_\rho \in L^0_\delta(\mathbb{R}^n, Z)$ *invertible for* $|\rho|$ *large so that*

$$\forall\, \varphi_1 \in C_0^\infty(\mathbb{R}^n), \exists \varphi_j \in C_0^\infty(\mathbb{R}^n), j = 2, 3, 4$$

so that

$$\varphi_1 \widetilde{P}_\rho A_\rho = (\varphi_1 B_\rho \varphi_2)\Lambda_\rho^{-1}(\Delta_\rho I + \varphi_3 R^{(-N)} \varphi_4) \tag{6.11}$$

with $R^{-N} \in L_\delta^{-N}(\mathbb{R}^n, Z)$, *and* $\forall s \in \mathbb{R}$

$$\|\varphi_3 R^{(-N)} \varphi_4\|_{s,s} \leq C_s |\rho|^{-N}.$$

The last estimate follows from Theorem 6, since the kernel of $\phi_3 R^{-N} \phi_4$ has compact support independent of ρ. Then to find solutions of $P_\rho v_\rho = 0$ in compact sets, it is enough to find solutions of

$$(\Delta_\rho I + \varphi_3 R^{(-N)} \varphi_4) w_\rho = 0. \tag{6.12}$$

Then $\Lambda_\rho^{-1} A_\rho w_\rho$ solves $P_\rho v_\rho = 0$ on compact sets (take $\varphi_1 = 1$ on the compact set).

The proof of this result for the case $\delta = 0$ is given in [N-U I]. We will write in this case $L_0^m(\mathbb{R}^n, Z) =: L^m(\mathbb{R}^n)$ and $S_0^m(U, Z) =: S^m(U, Z)$. We'll just say a few words about the proof which is quite technical. The main problem is to construct A_ρ, B_ρ near the characteristic variety (i.e. the set of zeros of $\widetilde{r}_\rho$ with $\widetilde{r}_\rho$ as in Example 1). This is because away from the characteristic variety P_ρ and Δ_ρ are elliptic and therefore invertible modulo elements of $L^{-N}(\mathbb{R}^n, Z)$ for all $N \in \mathbb{N}$ and it is therefore easy to construct the intertwining operators A_ρ, B_ρ in that case. The characteristic variety is given by

$$r_\rho^{-1}(0) = \{(x, \xi) \in \mathbb{R}^{2n}; \operatorname{Re}\rho \cdot \xi = 0, |\xi + \operatorname{Im}\rho|^2 = |\operatorname{Im}\rho|^2\}$$

where

$$r_\rho(\xi) = (|\xi|^2 + |\rho|^2)^{-\frac{1}{2}}(-|\xi|^2 + 2i\rho \cdot \xi).$$

Near the characteristic variety we take $A_\rho = B_\rho$.

So we are looking for $A_\rho \in L^0(\mathbb{R}^n, Z)$ such that

$$\widetilde{P}_\rho A_\rho = A_\rho \widetilde{\Delta}_\rho \bmod L^{-N}(\mathbb{R}^n, Z)$$

i.e.

$$(\widetilde{\Delta}_\rho + \widetilde{P}_\rho^{(1)}(x, D)) A_\rho = A_\rho \widetilde{\Delta}_\rho \bmod L^{-N}(\mathbb{R}^n, Z).$$

We proceed inductively. We choose

$$A_\rho = \sum_{j=0}^{M} A_\rho^{(j)}, A_\rho^{(j)} \in L^{-j}(\mathbb{R}^n, Z)$$

Let $\sigma_j(A_\rho^{(j)})$ be the principal symbol of $A_\rho^{(j)}$. Then by the calculus of pseudodifferential operators depending on a parameter we need to solve

$$H_{r_\rho}\sigma_0(A_\rho^{(0)}) + \sigma_0(\widetilde{P}_\rho^{(1)})\sigma_0(A_\rho^{(0)}) = 0 \tag{6.13}$$

and

$$H_{r_\rho}(\sigma_j(A_\rho^{(j)}) + \sigma_0(\widetilde{P}_0^{(1)})\sigma_j(A_\rho^{(M)}) = g_j(x, \xi)$$

with $g_j \in S^{-j}(\mathbb{R}^n, Z)$ so that

$$\widetilde{P}_\rho A_\rho = A_\rho \widetilde{\Delta}_\rho \bmod L^{-M}(\mathbb{R}^n, Z).$$

(Recall that this is all done near the characteristic variety). We note that

$$H_{r_\rho} = L_{1,\rho} + iL_{2,\rho}$$

with $L_{1,\rho}, L_{2,\rho}$ real-valued vector fields $\mathbb{R}^{2n}$ so that $[L_{1,\rho}, L_{2,\rho}] = 0$. Therefore H_{r_ρ} can be reduced to a Cauchy-Riemann equation in two variables of the form $\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}$. In fact one can write down an explicit change of variables in (ξ, ρ) to accomplish this (see [N-U]). We also give conditions at ∞ in on A_ρ to guarantee that it is invertible. Namely in the coordinates in which the vector field H_{r_ρ} is the Cauchy-Riemman equation we require, for $-1 < \alpha < 0$, that $\sigma(A_\rho) - I \in L^2_\alpha$ where I denotes the identity matrix.

To prove Theorem 5.2 with less regularity in the magnetic potentials Tolmasky ([To]) constructed complex geometrical optics solutions with the coefficients of $P^{(1)}(x, D)$ in $C^{2/3+\epsilon}(\overline{\Omega})$ for any ϵ positive. Using techniques from the theory of pseudodifferential operators with non smooth symbols ([Bo], [C-M], [T]) one can decompose a non-smooth symbol into a smooth symbol plus a less smooth symbol but of lower order. We describe below more precisely this result. First we introduce some notation. $C^s_*(\mathbb{R}^n)$ will denote the Zygmund class.

Definition 6.5 *Let* $\delta \in [0, 1]$*:*

(a) $p_\rho(x, \xi) \in C^s_* S^m_{1,\delta,\rho}(\mathbb{R}^n)$ *if and only if*

$$|D^\alpha_\xi p_\rho(x, \xi)| \leq C_\alpha((1 + |\xi|^2 + |\rho|^2)^{\frac{1}{2}})^{m-|\alpha|}$$

and

$$\|D_\xi^\alpha p_\rho(\cdot,\xi)\|_{C_*^s} \leq C_\alpha \ ((1+|\xi|^2+|\rho|^2)^{\frac{1}{2}})^{m-|\alpha|+s\delta}$$

for any $\alpha \in \mathbf{Z}_+^n$.

(b) $p_\rho(x,\xi) \in C^s S^m_{1,\delta,\rho}(\mathbb{R}^n)$ *if, the conditions on (a) are satisfied and additionally:*

$$\|D_\xi^\alpha p_\rho(\cdot,\xi)\|_{C^j} \leq C_\alpha((1+|\xi|^2+|\rho|^2)^{\frac{1}{2}})^{m-|\alpha|+j\delta}$$

for any $\alpha \in \mathbf{Z}_+^n$ *and* $\forall\ j \in \mathbb{N}$ *such that* $0 \leq j \leq s$.

Proposition 6.2 *Let* $p_\rho(x,\xi) \in C_*^s S^m_{1,0,\rho}$, *then we can write for any* δ *so that* $0 \leq \delta < 1$:

$$p_\rho(x,\xi) = p_\rho^\sharp(x,\xi) + p_\rho^b(x,\xi) \tag{6.14}$$

where

$$p_\rho^\sharp(x,\xi) \in \ S^m_{1,\delta,\rho} \tag{6.15}$$

and

$$p_\rho^b(x,\xi) \in C_*^{s-t} S^{m-t\delta}_{1,0,\rho} \quad s,\ s-t>0 \tag{6.16}$$

Let $p_\rho(x,D)$, $p_\rho^\sharp(x,D)$, $p_\rho^b(x,D)$ denote the corresponding operators associated to $p_\rho(x,\xi)$, $p_\rho^\sharp(x,\xi)$, $p_\rho^b(x,\xi)$, respectively. Then we have the following estimates which are proved using a Littlewood-Paley decomposition of the phase space depending on the parameter ρ.

Theorem 6.1 *Let* $p_\rho(x,\xi) \in C_*^r S^m_{1,0,\rho}(\mathbb{R}^n)$. *Then*

$$p_\rho(x,D): \ H^{s+m,p}(\mathbb{R}^n) \longrightarrow H^{s,p}(\mathbb{R}^n)$$

with

$$\|p_\rho(x,D)\|_{s+m,s} \leq C((1+|\rho|^2)^{\frac{1}{2}})^{s+m} \tag{6.17}$$

where $0 < s < r$, $p \in (1,\infty)$ *and* $\| \ \|_{s+m,s}$ *denotes the operator norm between Sobolev spaces.*

Now we describe how to construct complex geometrical optics solution of

$$P(x,D) = \Delta + P^{(1)}(x,D) \tag{6.18}$$

Using Theorem (6.3) (for the sake of exposition we will eliminate all the cut-off functions) we can find operators $A_\rho, B_\rho \in L^0_\delta(\mathbb{R}^n, Z)$ such that A_ρ, B_ρ are invertible for large ρ and

$$(\Delta_\rho + N_\rho^\sharp)A_\rho = B_\rho(\Delta_\rho + C_\rho) \tag{6.19}$$

with $C_\rho \in L^0_\delta(\mathbb{R}^n, Z)$ where $P^{(1)}_\rho(x, D) = N^\sharp_\rho(x, D) + N^\flat_\rho(x, D)$ using the decomposition (6.14). Then

$$(6.20) \qquad \begin{aligned} (\Delta_\rho + P_\rho(x, D))A_\rho &= B_\rho(\Delta_\rho + C_\rho) + N^\flat_\rho(x, D)A_\rho \\ &= B_\rho(\Delta_\rho + C_\rho + B_\rho^{-1} N^\flat_\rho(x, D)A_\rho) \end{aligned}$$

Using the estimate (6.17) and the estimates for the operators A_ρ, B_ρ implied by Theorem 6.2 we conclude, the following estimate under the regularity assumption that the coefficients of the first order term are in $C^{2/3+\epsilon}$ with $\epsilon > 0$

$$(6.21) \qquad \|(C_\rho + B_\rho^{-1} N^\flat_\rho(x, D)A_\rho)\Delta_\rho^{-1}\|_{L^2(\Omega), L^2(\Omega)} \leq C|\rho|^{-\beta}$$

for some $\beta = \beta(\epsilon) > 0$.

7 Anisotropic conductors

In sections 0-4 we considered isotropic conductivities, i.e. the electrical properties of Ω don't depend of direction. Examples of anisotropic media are muscle tissue. In this section we consider the inverse conductivity problem for anisotropic medium. The problem is well understood in two dimensions. Using isothermal coordinates ([A]) one can in fact reduce, by a change of variables, the anisotropic conductivity equation to an isotropic one and therefore one can apply the two dimensional results of section 4. Of course this is not available in dimension $n > 2$. In fact in this case the problem is equivalent to the problem of determining a Riemannian metric from the Dirichlet to Neumann map associated to the Laplace-Beltrami operator [L-U].

In this section we will consider the case of a quasilinear anisotropic conductivity. We outline recent results [Su-U III] proving identical results to the linear case. One needs to go further than the linearization procedure of Lemma 3.1 for isotropic non-linear conductivities. In fact we show that one can reduce the problem question about the density of product of solutions for the linear anisotropic conductivities, by using a second linearization.

We assume that $\gamma(x, t) \in C^{1,\alpha}(\overline{\Omega} \times \mathbb{R})$ be a symmetric, positive definite matrix function satisfying

$$(7.1) \qquad \gamma(x, t) \geq \epsilon_T I, \quad (x, t) \in \overline{\Omega} \times [-T, T], T > 0,$$

where $\epsilon_T > 0$ and I denotes the identity matrix.

It is well known (see e.g. [G-T]) that, given $f \in C^{2,\alpha}(\overline{\Omega})$, there exists a unique solution of the boundary value problem

$$(7.2) \qquad \begin{cases} \nabla \cdot (\gamma(x, u)\nabla u) &= 0 \qquad \text{in } \Omega, \\ u\big|_{\partial\Omega} &= f. \end{cases}$$

We define the Dirichlet to Neumann map $\Lambda_\gamma : C^{2,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$ as the map given by

$$\Lambda_\gamma : f \to \nu \cdot \gamma(x, f)\nabla u\Big|_{\partial\Omega}, \tag{7.3}$$

where u is the solution of (7.2) and ν denotes the unit outer normal of $\partial\Omega$.

Physically, $\gamma(x, u)$ represents the (anisotropic, quasilinear) conductivity of Ω and $\Lambda_\gamma(f)$ the current flux at the boundary induced by the voltage f.

We study the inverse boundary value problem associated to (7.2): how much information about the coefficient matrix γ can be obtained from knowledge of the Dirichlet to Neumann map Λ_γ?

The uniqueness, however, is false in the case where γ is a general matrix function as it is also in the linear case [K-V III]: if $\Phi : \overline{\Omega} \to \overline{\Omega}$ is a smooth diffeomorphism which is the identity map on $\partial\Omega$, and if we define

$$(\Phi_*\gamma)(x, t) = \frac{(D\Phi)^T \gamma(\cdot, t)(D\Phi)}{|D\Phi|} \circ \Phi^{-1}(x), \tag{7.4}$$

then it follows that

$$\Lambda_{\Phi_*\gamma} = \Lambda_\gamma, \tag{7.5}$$

where $D\Phi$ denotes the Jacobian matrix of Φ and $|D\Phi| = \det(D\Phi)$.

The main results of [Su-U III] concern with the converse statement. We have

Theorem 7.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $C^{3,\alpha}$ boundary, $0 < \alpha < 1$. Let γ_1 and γ_2 be quasilinear coefficient matrices in $C^{2,\alpha}(\overline{\Omega} \times \mathbb{R})$ such that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then there exists a $C^{3,\alpha}$ diffeomorphism $\Phi : \overline{\Omega} \to \overline{\Omega}$ with $\Phi\big|_{\partial\Omega} =$ identity, such that $\gamma_2 = \Phi_*\gamma_1$.*

Theorem 7.2 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded simply connected domain with real-analytic boundary. Let γ_1 and γ_2 be real-analytic quasilinear coefficient matrices such that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Assume that either γ_1 or γ_2 extends to a real-analytic quasilinear coefficient matrix on $\mathbb{R}^n$. Then there exists a real-analytic diffeomorphism $\Phi : \overline{\Omega} \to \overline{\Omega}$ with $\Phi\big|_{\partial\Omega} =$ identity, such that $\gamma_2 = \Phi_*\gamma_1$.*

Theorems 7.1 and 7.2 generalize all known results for the linear case ([S-U IV]). In this case and $n = 2$, with a slightly different regularity assumption, Theorem 7.1 follows using a reduction theorem of Sylvester [S], using isothermal coordinates, and Theorem 4.1 for the isotropic case.

In the linear case and $n \geq 3$, Theorem 7.2 is a consequence of the work of Lee and Uhlmann [L-U], in which they discussed the same problem on real-analytic Riemannian manifolds. The assumption that one of the coefficient matrices can be extended analytically to $\mathbb{R}^n$ can be replaced by a convexity assumption on the Riemannian metrics associated to the coefficient matrices. Thus Theorem 7.2 can also be stated under this assumption, which we omit here. We mention that, in the linear case, complex geometrical optics solutions have not been constructed for the Laplace-Beltrami operator in dimensions $n \geq 3$. The proof of Theorem 2.1 in the linear case follows a different approach.

A. Linearization The proof of linearization Lemma 3.1 is also valid in the anisotropic case. We shall use γ^t to denote the function of x obtained by freezing t in $\gamma(x,t)$.

Under the assumptions of Theorem 1.1., using Lemma 3.1 we have that

$$\Lambda_{\gamma_1^t} = \Lambda_{\gamma_2^t}, \quad \forall\, t \in \mathbb{R}. \tag{7.6}$$

Since Theorems 7.1 and 7.2 hold in the linear case, it follows that, there exists a diffeomorphism Φ^t, which is in C^3_α when $n = 2$ and is real-analytic when $n \geq 3$, and the identity at the boundary such that

$$\gamma_2^t = \Phi^t_* \gamma_1^t. \tag{7.7}$$

It is proven in [Su- U III] that Φ^t is uniquely determined by γ_l^t, and thus by γ_l, $l = 1, 2$. We then obtain a function

$$\Phi(x,t) = \Phi^t(x) : \overline{\Omega} \times \mathbb{R} \to \overline{\Omega} \times \mathbb{R}, \tag{7.8}$$

which is in $C^{3,\alpha}(\overline{\Omega})$ for each fixed t in dimension two and real analytic in dimension $n \geq 3$. It is also shown in [Su -U I] that Φ is also smooth in t. More precisely we have, in every dimension $n \geq 2$, that $\frac{\partial \Phi}{\partial t} \in C^{2,\alpha}(\overline{\Omega})$.

In order to prove Theorems 7.1 and 7.2, we must then show that Φ^t is independent of t. Without loss of generality, we shall only prove

$$\left.\frac{\partial \Phi}{\partial t}\right|_{t=0} = 0 \qquad \text{in } \overline{\Omega}. \tag{7.9}$$

It is easy to show, using the invariance (7.5) that we may assume that

$$\Phi(x,0) \equiv x, \text{that is,} \quad \Phi^0 = \text{ identity}. \tag{7.10}$$

Let us fix a solution $u \in C^{3,\alpha}(\overline{\Omega})$ of

$$\nabla \cdot A \nabla u = 0, \quad u|_{\partial\Omega} = f, \tag{7.11}$$

where we denote $A = \gamma_1^0 = \gamma_2^0$.

For every $t \in \mathbb{R}$ and $l = 1, 2$, we solve the boundary value problem (7.11) with γ^t replaced by γ_l^t. We obtain a solution $u_{(l)}^t$:

$$\left\{ \begin{array}{rcll} \nabla \cdot \gamma_l^t \nabla u_{(l)}^t & = & 0 & \text{in } \Omega \\ u_{(l)}^t \Big|_{\partial\Omega} & = & f, & \qquad l = 1, 2. \end{array} \right. \tag{7.12}$$

It is easy to check that

$$u_{(1)}^t(x) = u_{(2)}^t(\Phi^t(x)), \qquad x \in \overline{\Omega}.$$

Differentiating this last formula in t and evaluating at $t = 0$ we obtain

$$\left(\frac{\partial u_{(1)}^t}{\partial t} - \frac{\partial u_{(2)}^t}{\partial t} \right) \Bigg|_{t=0} - X \cdot \nabla u = 0, \qquad x \in \overline{\Omega}, \tag{7.13}$$

where

$$X = \frac{\partial \Phi^t}{\partial t} \bigg|_{t=0}. \tag{7.14}$$

It is easy to show that $X \cdot \nabla u = 0$ for every solution of (7.11) implies $X = 0$. So we are reduced to prove

$$\left(\frac{\partial u_{(1)}^t}{\partial t} - \frac{\partial u_{(2)}^t}{\partial t} \right) \Bigg|_{t=0} = 0. \tag{7.15}$$

Using (7.12) we get

$$\nabla \cdot (\gamma_1(x,t) \nabla u_{(1)}^t) - \nabla \cdot (\gamma_2(x,t) \nabla u_{(2)}^t) = 0. \tag{7.16}$$

Differentiating (7.16) in t at $t = 0$ we conclude

$$\nabla \cdot \left[\left(\frac{\partial \gamma_1}{\partial t} - \frac{\partial \gamma_2}{\partial t} \right) \bigg|_{t=0} \nabla u \right] + \nabla \cdot \left[A \nabla \left(\frac{\partial u_{(1)}^t}{\partial t} - \frac{\partial u_{(2)}^t}{\partial t} \right) \Bigg|_{t=0} \right] = 0. \tag{7.17}$$

We claim that to prove (7.15) it is enough to show that

$$\nabla \cdot \left[\left(\frac{\partial \gamma_1}{\partial t} - \frac{\partial \gamma_2}{\partial t} \right) \bigg|_{t=0} \nabla u \right] = 0. \tag{7.18}$$

This is the case since we get from (7.17) and (7.18)

$$\nabla \cdot \left[A\nabla \left(\frac{\partial u^t_{(1)}}{\partial t} - \frac{\partial u^t_{(2)}}{\partial t} \right)\Bigg|_{t=0} \right] = 0.$$

The claim now follows since the operator $\nabla \cdot A\nabla : \overset{\circ}{H}{}^2(\Omega) \cap H^1(\Omega) \to L^2(\Omega)$ is an isomorphism and therefore

$$\left(\frac{\partial u^t_{(1)}}{\partial t} - \frac{\partial u^t_{(2)}}{\partial t} \right)\Bigg|_{t=0}\Bigg|_{\partial\Omega} = 0.$$

B. Second linearization and products of solutions

In order to show (7.18) we now study the second linearization. We introduce, for every $t \in \mathbb{R}$, the map $K_{\gamma,t} : C^{2,\alpha}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ which is defined implicitly as follows (see [Su I]): for every pair $(f_1, f_2) \in C^{2,\alpha}\partial\Omega) \times C^{2,\alpha}(\partial\Omega)$,

$$\int_{\partial\Omega} f_1 K_{A,t}(f_2) ds = \int_{\Omega} \nabla u_1 \frac{\partial A}{\partial t} \nabla u_2^2 dx \tag{7.19}$$

with $u_l, l = 1, 2$, as in (7.12) with f replaced by $f_l, l = 1, 2$. We have

Proposition 7.1 ([Su I]) *Let $\gamma(x,t)$ be a positive definite symmetric matrix in $C^2(\overline{\Omega} \times \mathbb{R})$, satisfying (7.1). Then for every $f \in C^{2,\alpha}(\partial\Omega)$ and $t \in \mathbb{R}$,*

$$\lim_{s \to 0} \left\| \frac{1}{s} \left[\frac{1}{s} \Lambda_A(t + sf) - \Lambda_{A^t}(f) \right] - K_{A,t}(f) \right\|_{H^{\frac{1}{2}}(\partial\Omega)} = 0.$$

Under the assumptions of Theorems 7.1 and 7.2, using Proposition 7.1 with $t = 0$, we obtain

$$K_{\gamma_{1,0}}(f) = K_{\gamma_{2,0}}(f), \qquad \forall\ f \in C^{3,\alpha}(\partial\Omega).$$

Thus, by (7.19) we have

$$\int_{\Omega} \nabla u_1 \left. \frac{\partial \gamma_1}{\partial t} \right|_{t=0} \nabla u_2^2 dx = \int_{\Omega} \nabla u_1 \left. \frac{\partial \gamma_2}{\partial t} \right|_{t=0} \nabla u_2^2 dx, \tag{7.20}$$

with u_1, u_2 solutions of (7.12). By writing

$$B = \left(\frac{\partial \gamma_1}{\partial t} - \frac{\partial \gamma_2}{\partial t} \right)\Bigg|_{t=0} \tag{7.21}$$

and replacing in (7.20) u_1 by u and u_2^2 by $(u_1+u_2)^2 - u_1^2 - u_2^2$, we obtain

$$\int_\Omega \nabla u \cdot B(x) \nabla (u_1 u_2) dx = 0 \tag{7.22}$$

with u, u_1 and u_2 solutions of (7.12).

To continue from (7.22), we need the following two lemmas.

Lemma 7.1 *Let $h(x) \in C^1(\overline{\Omega})$ be a vector-valued function. If*

$$\int_\Omega h(x) \nabla (u_1 u_2) dx = 0$$

for arbitrary solutions u_1 and u_2 of (7.12) , then $h(x)$ lies in the tangent space $T_x(\partial\Omega)$ for all $x \in \partial\Omega$.

Lemma 7.2 *Let $A(x)$ be a positive definite, symmetric matrix in $C^{2,\alpha}(\overline{\Omega})$. Define*

$$D_A = \mathrm{Span}_{L^2(\Omega)}\{uv; u, v \in C^{3,\alpha}(\overline{\Omega}), \nabla \cdot A\nabla u = \nabla \cdot A \nabla v = 0\}.$$

Then the following are valid:

(a)If $l \in C^\omega(\overline{\Omega})$ and $l \perp D_A$, then $l = 0$ in $\overline{\Omega}$

(b)If $n = 2$, then $D_A = L^2(\Omega)$.

Now we finish the proof of (7.18) concluding the proofs of Theorems 7.1 and 7.2.

By Lemma 7.1 we have that $\nu \cdot B(x)\nabla u \equiv 0$ in $\partial\Omega$. Integrating by parts in (7.22), we obtain

$$\int_\Omega [\nabla \cdot B(x) \nabla u] u_1 u_2 dx = 0. \tag{7.23}$$

We now apply Lemma 7.1 to (7.23). If $n \geq 3$, we have that γ_1 and γ_2 are real-analytic on $\overline{\Omega} \times \mathbb{R}$. Thus $B \in C^\omega(\overline{\Omega})$. Since the solutions u solves an elliptic equation with a real-analytic coefficient matrix, we have that u is analytic in Ω. If u is analytic on $\overline{\Omega}$, we can conclude from Lemma 7.2 that

$$\nabla \cdot (B(x) \nabla u) = 0, \qquad x \in \overline{\Omega}. \tag{7.24}$$

We shall prove that (7.24) holds independent of whether u is analytic up to $\partial\Omega$ or not. This is due to the Runge approximation property of the equation. Using the assumptions of Theorem 7.1, we extend A analytically to a slightly larger domain $\underset{\sim}{\Omega} \supset \overline{\Omega}$. For any solution $u \in C^{3,\alpha}(\overline{\Omega})$ and an open subset $\mathcal{O}$ with $\overline{\mathcal{O}} \subset \Omega$, we can find a sequence of solutions $\{u_m\} \subset C^{\omega}(\widetilde{\Omega})$, which solves (7.22) on $\widetilde{\Omega}$, and $u_m\Big|_{\mathcal{O}_1} \underset{m\to\infty}{\longrightarrow} u\Big|_{\mathcal{O}_1}$ in the L^2 sense, where $\overline{\mathcal{O}}_1 \subset \Omega$, $\overline{\mathcal{O}} \subset \mathcal{O}_1$. By the local regularity theorem of elliptic equations this convergence is valid in $H^2(\mathcal{O})$. Since (7.24) holds with $u = u_m$, letting $m \to \infty$ yields the desired result for u on $\mathcal{O}$. Thus (7.22) holds. If $n = 2$, Lemma 7.1(b) implies that $\nabla \cdot (B(x)\nabla u) = 0$ for any solution $u \in C^{3,\alpha}(\overline{\Omega})$.

The proof of Lemma 7.1 follows an argument of Alessandrini [Al II], which relies on the use of solutions with isolated singularities. It turns out that in our case, only solutions with Green's function type singularities are sufficient in the case $n \geq 3$, while in the case $n = 2$, solutions with singularities of higher order must be used. There are additional difficulties since we are dealing with a vector function h. We refer the readers to [Su-U III] for details.

The proof of part (a) of Lemma 7.2 follows the proof of Theorem 1.3 in [Al] (which also follows the arguments of [K-V I]). Namely, one constructs solutions u of (7.11) in a neighborhood of Ω with an isolated singularity of arbitrary given order at a point outside of Ω. We then plug this solution into the identity

$$\int_{\Omega} l u^2 dx = 0.$$

By letting the singularity of u approach to a point x in $\partial\Omega$, one can show that any derivative of l must vanish on x and thus by the analyticity of l, $l \equiv 0$ in $\overline{\Omega}$. For more e details see [Su-U III].

To prove the part (b) of Lemma 7.2, we first reduce the problem to the Schrödinger equation.

Using isothermal coordinates (see [A]), there is a conformal diffeomorphism $F : (\overline{\Omega}, g) \to (\overline{\Omega}', e)$, where g is the Riemannian metric determined by the linear coefficient matrix A with $g_{ij} = A_{ij}^{-1}$. One checks that F transforms the operator $\nabla \cdot A \nabla$ (on Ω) to an operator $\nabla \cdot A' \nabla$ (on Ω') with A' a scalar matrix function $\beta(x)I$. Therefore the proof of the part (b) is reduced to the case where $A = \beta I$, with $\beta(x) \in C^{2,\alpha}(\overline{\Omega})$. By approximating by smooth solutions, we see that the $C^{3,\alpha}$ smoothness can be replaced by H^2 smoothness. Thus we have reduced the problem to showing that

$$D_\beta = \mathrm{Span}_{L^2}\{uv; u, v \in H^2(\Omega); \nabla \cdot \beta \nabla u = \nabla \cdot \beta \nabla v = 0\} = L^2(\Omega).$$

We make one more reduction by transforming, as in section 2, the equation $\nabla \cdot \beta \nabla u = 0$ to the Schrödinger equation

$$\Delta v - qv = 0$$

with

$$u = \beta^{-\frac{1}{2}} v, q = \frac{\Delta\sqrt{\beta}}{\sqrt{\beta}} \in C^{\alpha}(\overline{\Omega}). \tag{7.25}$$

This allows us to reduce the proof to showing that

$$D_q = \text{Span}_{L^2}\{v_1 v_2; v_i \in H^2(\Omega), \Delta v_i - q v_i = 0, i = 1, 2\} = L^2(\Omega) \tag{7.26}$$

for potentials q of the form (7.25).

Statement (7.26) was proven by Novikov [No II]. In [Su-U III] it was shown that it is enough to use the Proposition below which is valid for any potential $q \in L^\infty(\Omega)$. This result uses some of the techniques of [Su -U I,II] similar to the proof of Theorem 4.6.

Proposition 7.2 *Let $q \in L^\infty(\Omega), n = 2$. Then D_q has a finite codimension in $L^2(\Omega)$.*

It is an interesting open question whether $D_q = L^2(\Omega)$ in the two dimensional case.

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