# APPLICATIONS OF THE STAR-K TOOL

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ABSTRACT. The purpose of this paper is to develop the useful star-K tool and apply it to recovery questions. We start by developing the star-K, and then give some basic examples. Some more complex examples are given. Specifically, the star-K is used to effectively explain the nature of $2-1$ networks, which have gone without clear explanations for far too long. We address how star-K's can shed light on the relationship between determinants and connections in the graph, and finally pose some open questions.

## CONTENTS



## 1. INTRODUCTION

The work in [3] outlines a functionally complete solution to the inverse problem it proposes in the circular planar case. The problem, stated in a few sentences, is this. We start with a graph, and then think about it like an electrical network in what seems to be a pretty natural way. That is, we start with a set of points,

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call them nodes, and a set of connections between those nodes, call them edges. We then designate some of those nodes as being special nodes, and we call them boundary nodes; we might as well call the left over nodes interior nodes. Finally, we assign to each connection between nodes a certain numerical value. We insist that these numerical values be positive, and call them conductivities. Now, thinking of this as an electrical system (as of wires and so forth) we write down what it means for Kirchoff's law to be satisfied at all the interior nodes. Since we are thinking of the boundary nodes as the sources and sinks in the network, we don't want to enforce Kirchoff's law there. Finally, we ask the following question. When and how can we determine the conductivities assigned to each edge in the graph, by applying voltages at boundary nodes, and measuring the resulting current, the response, at the other boundary nodes? This is the inverse problem.

As I mentioned above this inverse problem has been essentially exhausted in a specific case. "Circular planar" means that all the boundary nodes fall on a circle and the whole rest of the network fits inside that circle. Yet, many networks are not of this type. For example, one can not draw four points, boundary nodes, on circle and connect every pair by non-overlapping lines without leaving the circle. We would call this network $K_4$, the complete graph on 4 boundary nodes. It is complete in the sense that any two boundary nodes are connected by an edge. There is very little understood with regard to the inverse problem in the case of an arbitrary graph. With that as motivation, in this paper we will derive and use star-K transformations, a tool for examining arbitrary networks. Whether these techniques can be generalized to form a general recovery algorithm is examined in [1], and the author of that paper as well as Nick Addington contributed a great deal to the ideas and results in this paper.

Lastly, as a brief history I should mention that a motivating example for this technique is the remarkable light that it can shed on known $2-1$ networks. These networks, originally proposed in [2], were so boggling and intriguing that I could not stop working on them. The work paid off; The infamous triangle-in-triangle network has never been so clearly understood. In general, many of the cases of annular graphs exposed in [2] have been examined using this tool to various degrees of success.

## 2. The Star-K Transformation with 4 Boundary Nodes

In this section we will examine the Star-K transformation for the case of four boundary nodes. The star with four nodes is also called the plus. $K_n$ is shorthand for the complete graph, a graph with $n$ boundary nodes, no interior nodes, where every node is joined to every other by an edge.

On a star the current response at boundary node $i$ due to a unit voltage at node $j$ is easily written down by computing the voltage at the only interior node by the weighted average property. Knowing this, the current response is simply the product of the two conductors joining nodes $i$ and $j$ divided by the sum of the conductors around the interior node. This sum will often be abbreviated $\sigma$. Now we talk about transforming the star into a network on a complete graph that will still have the same current response for every pair of boundary nodes.

When we transform to a K we necessarily pick up some algebraic relations on the conductances. These relations have previously been thought of as determinants

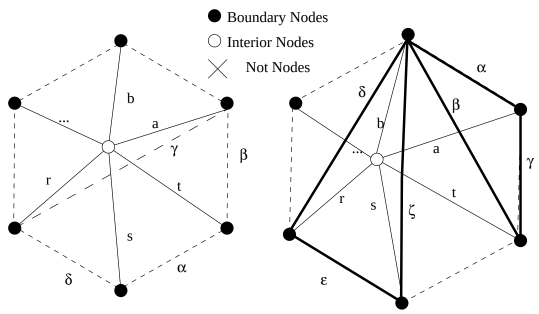


FIGURE 1. quadrilateral and Triangle Conditions

which are zero in the response matrix, see section 4, due to the total lack of two connections in the star, but [1] has a geometric interpretation that is quite clear and useful. That is, <u>the products of opposites sides of a quadrilateral in a K that came from a star are equal</u>. We can quickly prove this with what we have stated thus far. To say

$$\alpha\gamma = \beta\delta \tag{1}$$

is equivalent to saying

$$\frac{st}{\sigma}\frac{ar}{\sigma} = \frac{ta}{\sigma}\frac{rs}{\sigma}.$$

If we call (1) the quadrilateral condition we can rightly call (2) the triangle condition. For any K coming from a star, the product of the legs divided by the base is a constant for any pair of triangles sharing an vertex. The proof of this is quite straightforward as well. We simply remark that

$$\frac{\delta\zeta}{\epsilon} = \frac{\alpha\beta}{\gamma} \tag{2}$$

is equivalent to

$$\frac{br}{\sigma}\frac{bs}{\sigma}\frac{\sigma}{rs} = \frac{ba}{\sigma}\frac{bt}{\sigma}\frac{\sigma}{at}.$$

We can also write down the formula for transforming a star back into a K. If $\gamma_i$ is the conductor with boundary node $i$ in the star, $\Sigma_i$ is the sum of the conductors around node $i$ in the K, and $K_{ij}$ is the conductivity on the corresponding edge in the complete graph it is easy to show $\gamma_i$ is given by the formula

$$\gamma_i = \Sigma_i + \frac{K_{ij}K_{ik}}{K_{jk}}. \tag{3}$$

For the case of a four node K-star the transformation is shown below. Note: Roman letters are on the star, Greek on the K, as in the diagram.

$$a = \alpha + \gamma + \zeta + \frac{\alpha\zeta}{\delta} \qquad b = \alpha + \delta + \beta + \frac{\beta\delta}{\epsilon} \tag{4}$$

$$c = \beta + \gamma + \epsilon + \frac{\beta\gamma}{\alpha} \qquad d = \zeta + \delta + \epsilon + \frac{\zeta\delta}{\epsilon} \tag{5}$$

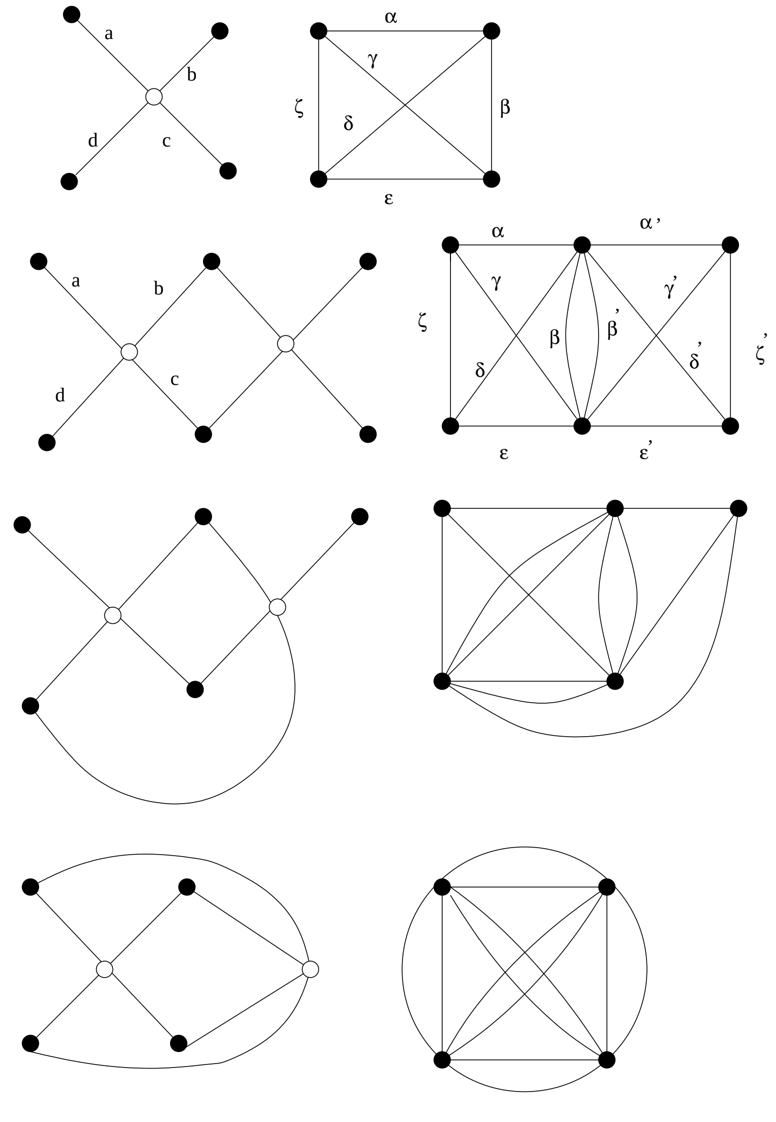


FIGURE 2. The Plus-K and a Few Applications

We can also write down the quadrilateral relations,

$$\alpha\epsilon = \gamma\delta \quad \beta\zeta = \alpha\epsilon \quad \alpha\epsilon = \gamma\delta. \tag{6}$$

When the quadrilateral conditions are satisfied these equations let us quickly recover a plus from a $K_4$ as in Figure 2. Figure 2 also shows two plus-graphs joined at two boundary nodes. We see that when we make the transformation to the $K_4$ on both pluses we get one parallel edge. However, this edge can be eliminated using the quadrilateral condition which shows that the graph can still be recovered.

The third pair of graphs in Figure 2, is two pluses joined at three boundary nodes. This is a more interesting case because the graph is not circular planar so we cannot use our circular planar tools to examine it. When we transform both pluses into K's we get three parallel edges. Clearly then, we can not use the quadrilateral condition to recover all the parallel edges and this graph is not recoverable. Furthermore, if we fix one parameter the others can be determined; from this we conclude that the solution space is one dimensional. The last pair shown is two pluses joined at all boundary nodes. In this case we get all six parallel edges. We have to specify four parameters before we can determine all the conductances so the solution space is four dimensional.

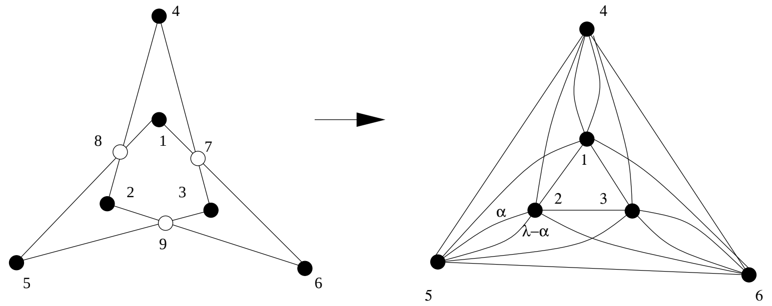


FIGURE 3. The Triangle in Triangle with Three plus-K Transformations

## 3. The Triangle in Triangle and other Two to One Graphs

Certain graphs have been found in previous years to have the special property that they could generate the same current response for exactly two sets of conductances. The "Triangle in Triangle" was the first of these and was described in [2] where an explicit quadratic formula for the conductances was found in terms of entries in the response matrix. After several pages of manipulations the terms of this quadratic formula were found to be approximately 20 entries in length. In this section we use the plus-K tool to derive a simpler quadratic equation and effectively explain the nature of this $2-1$ graph. As is done in Figure 3, first we draw the triangle in triangle graph to clearly show how it is a sum of three plus-graphs. Then we transform each plus into a K.

Entries in the response matrix, a matrix representation of the complete graph, are by convention negative. We will not use this convention. Instead, *let $\lambda_{ij}$ denote the conductivity, a positive quantity by definition, on the complete graph.* In this way $\lambda_{ij}$ is both the $ij^{th}$ entry in the response matrix, and the conductivity of the edge joining nodes $i$ and $j$ on the complete graph.

Now, refer to figure 3. Using the quadrilateral condition we can see that the parallel edges cannot be found within a single $K_4$ so we assign a parameter to one of the edges, say $\alpha$. Then the other edge must be $\lambda_{25}-\alpha$ so that the sum of the parallel edges is $\lambda_{25}$. The quadrilateral relation gives us one of the edges joining node 1 to node 4 as the product of edges 1,2 and 4,5 divided by $\alpha$, or $\frac{\lambda_{12}\lambda_{45}}{\alpha}$. As before, the other edge must be $\lambda_{14}-\frac{\lambda_{12}\lambda_{45}}{\alpha}$. In turn, we can solve for the edges joining node 3 to node 6 with the product of edges 1,3 and 4,6 divided by $\lambda_{14}-\frac{\lambda_{12}\lambda_{45}}{\alpha}$. So one edge is

$$\frac{\lambda_{13}\lambda_{46}}{\lambda_{14}-\frac{\lambda_{12}\lambda_{45}}{\alpha}} \text{ and the other edge is } \lambda_{36}-\frac{\lambda_{13}\lambda_{46}}{\lambda_{14}-\frac{\lambda_{12}\lambda_{45}}{\alpha}}.$$

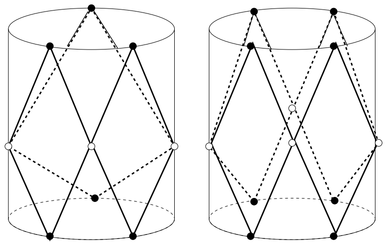

FIGURE 4. The Triangle-In-Triangle and Square-In-Square Graphs Embedded on the Cylinder

Applying the quadrilateral condition one final time gives us

$$\alpha = \lambda_{25} - \frac{\lambda_{23}\lambda_{56}}{\lambda_{36} - \frac{\lambda_{13}\lambda_{46}}{\lambda_{14} - \frac{\lambda_{12}\lambda_{45}}{\alpha}}}. \tag{7}$$

This quadratic equation can also be written in the more familiar form,

$$\alpha^2[\lambda_{13}\lambda_{46} - \lambda_{36}\lambda_{14}] + \alpha[\lambda_{14}\lambda_{23}\lambda_{56} + \lambda_{36}\lambda_{14}\lambda_{25} - \lambda_{36}\lambda_{12}\lambda_{46} - \lambda_{13}\lambda_{46}\lambda_{25}] + \lambda_{23}\lambda_{56}\lambda_{12}\lambda_{45} = 0.$$

In this form we can see that we do indeed have a quadratic because the coefficient of $\alpha^2$ is non-zero. The coefficient is non-zero for the following reason. If all the edges involved had come from a star, then by the results previously stated, $\lambda_{13}\lambda_{46} - \lambda_{36}\lambda_{14}$ would be zero. Yet, in our case $\lambda_{36}$ and $\lambda_{14}$ come from two stars. Hence, they both have a positive quantity added to the value they would have to be in order to make the term zero. Thus, the coefficient is not zero. For a further explanation of this see section 4.

This corresponds to the determinant of a unique connection which is present in the network. We expand on the relationship between connections and determinants in section 4.

### 3.1. The Locus of Degenerate Points.

The locus of degenerate points and other interesting characteristics of this graph have led to previous attempts to understand $2-1$ behavior. Many such attempts were made before the star-K tool was discovered so we will give some mention of these here. First, when the discriminant of this quadratic equals zero we get a locus of points where the conductances can be determined exactly from the response matrix. This discriminant is given by $D = b^2 - 4ac$, or

$$D = [\lambda_{14}\lambda_{23}\lambda_{56} + \lambda_{36}\lambda_{14}\lambda_{25} - \lambda_{36}\lambda_{12}\lambda_{46} - \lambda_{13}\lambda_{46}\lambda_{25}]^2 - 4[\lambda_{13}\lambda_{46} - \lambda_{36}\lambda_{14}][\lambda_{23}\lambda_{56}\lambda_{12}\lambda_{45}].$$

Another way to examine this behavior is from a topological point of view. If we take a parameterization of $\alpha$ as $\alpha = -\frac{b}{2a} + t\frac{D}{2a}$ with $t \in [-1, 1]$ and then compute the response matrix $\Lambda(t)$, then $\Lambda(t)$ forms a closed curve as $t$ varies.

### 3.2. Other Two to One and $2^n$ to One Graphs.

Other two to one graphs were also known to exist in previous years. However, without any precise tools to examine them, the algebra involved was very complex and in some cases misleading. For example, some networks that seem to have every property of a $2-1$ network in

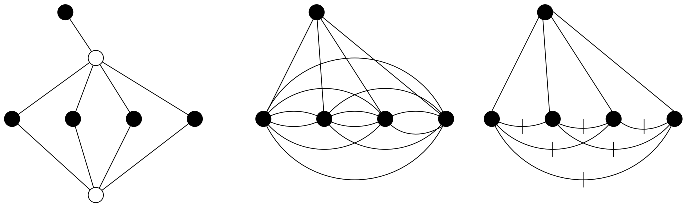

FIGURE 5. An Example Network and its Star-K Equivalent Graph

an algebraic sense can actually be shown using the star-K tool to be recoverable. However, we can now prove our original conjecture that an "n-gon in an n-gon graph" is $2-1$. An "n-gon in an n-gon graph" is a graph consisting of n plus-graphs joined at two boundary nodes such that they form a chain that loops back to the original plus-graph. This can be visualized as in figure 4 as n diamonds embedded on the cylinder. Annular graphs like those in [2] can in general can be embedded on the cylinder (no caps on the top or bottom) with their boundary nodes on the two boundary circles. Before we work on these, we should develop a complete definition of the response matrix so that we can apply, and in some cases derive, useful results from [3].

## 4. Connections and Determinants(In Progress)

Insert riveting exposition of the response matrix, and the relationships of sub-determinants thereof to connections in the graph. Emphasize star-K as way to understand this.

4.1. **The Response Matrix.** A complete discussion of the response matrix can be found in [3], but we will develop the ideas in a slightly different way so it may prove useful to read this section even if you are already familiar with the response matrix.

We define the response matrix to be the $n$ x $n$ matrix, where $n$ is the number of boundary nodes, such that the $ij^{th}$ entry denoted $\lambda_{ij}$ is the current that flows out of node $i$ due to a single unit voltage at node $j$. It can be shown that the response matrix is symmetric. The response matrix can also be calculated quickly using the Schur-Complement. The interested reader should refer to [3]. We will work out one example here to make solid our construction.

Consider the network shown in figure 3.2. Let $\sigma_6$ and $\sigma_7$ denote the sums of the conductances surrounding nodes 6 and seven respectively. Then, it is a simple extension of the work done in deriving the formula for $\lambda_{ij}$ in the case of star, as in section 2 to see that the response matrix, denoted $(\lambda_{ij})$ or $\Lambda$, is essentially given by

$$\begin{bmatrix} * & \frac{\gamma_{16}\gamma_{26}}{\sigma_6} & \frac{\gamma_{16}\gamma_{36}}{\sigma_6} & \frac{\gamma_{16}\gamma_{46}}{\sigma_6} & \frac{\gamma_{16}\gamma_{56}}{\sigma_6} \\ * & * & \frac{\gamma_{26}\gamma_{36}}{\sigma_6}+\frac{\gamma_{27}\gamma_{37}}{\sigma_7} & \frac{\gamma_{26}\gamma_{46}}{\sigma_6}+\frac{\gamma_{27}\gamma_{47}}{\sigma_7} & \frac{\gamma_{26}\gamma_{56}}{\sigma_6}+\frac{\gamma_{27}\gamma_{57}}{\sigma_7} \\ * & * & * & \frac{\gamma_{36}\gamma_{46}}{\sigma_6}+\frac{\gamma_{37}\gamma_{47}}{\sigma_7} & \frac{\gamma_{36}\gamma_{56}}{\sigma_6}+\frac{\gamma_{37}\gamma_{57}}{\sigma_7} \\ * & * & * & * & \frac{\gamma_{46}\gamma_{56}}{\sigma_6}+\frac{\gamma_{47}\gamma_{57}}{\sigma_7} \\ * & * & * & * & * \end{bmatrix}.$$

We have written *'s in certain places to communicate the fact that response matrix has a lot redundantly redundant information in it. Because the response matrix is symmetric we can find all the off diagonal entries from the information above. And because the total amount of current is conserved the diagonal entries are simply the negative of the row sums without the diagonal entries. The complete response matrix, which we are interested in, is shown below.

$$\begin{bmatrix} -\frac{\gamma_{16}(\sigma_6-\gamma_{16})}{\sigma_6} & \frac{\gamma_{16}\gamma_{26}}{\sigma_6} & \frac{\gamma_{16}\gamma_{36}}{\sigma_6} & \frac{\gamma_{16}\gamma_{46}}{\sigma_6} & \frac{\gamma_{16}\gamma_{56}}{\sigma_6} \\ \frac{\gamma_{16}\gamma_{26}}{\sigma_6} & -\frac{\gamma_{26}(\sigma_6-\gamma_{26})}{\sigma_6} - \frac{\gamma_{27}(\sigma_7-\gamma_{27})}{\sigma_7} & \frac{\gamma_{26}\gamma_{36}}{\sigma_6} + \frac{\gamma_{27}\gamma_{37}}{\sigma_7} & \frac{\gamma_{26}\gamma_{46}}{\sigma_6} + \frac{\gamma_{27}\gamma_{47}}{\sigma_7} & \frac{\gamma_{26}\gamma_{56}}{\sigma_6} + \frac{\gamma_{27}\gamma_{57}}{\sigma_7} \\ \frac{\gamma_{16}\gamma_{36}}{\sigma_6} & \frac{\gamma_{26}\gamma_{36}}{\sigma_6} + \frac{\gamma_{27}\gamma_{37}}{\sigma_7} & * & \frac{\gamma_{36}\gamma_{46}}{\sigma_6} + \frac{\gamma_{37}\gamma_{47}}{\sigma_7} & \frac{\gamma_{36}\gamma_{56}}{\sigma_6} + \frac{\gamma_{37}\gamma_{57}}{\sigma_7} \\ \frac{\gamma_{16}\gamma_{46}}{\sigma_6} & \frac{\gamma_{26}\gamma_{46}}{\sigma_6} + \frac{\gamma_{27}\gamma_{47}}{\sigma_7} & \frac{\gamma_{36}\gamma_{46}}{\sigma_6} + \frac{\gamma_{37}\gamma_{47}}{\sigma_7} & * & \frac{\gamma_{46}\gamma_{56}}{\sigma_6} + \frac{\gamma_{47}\gamma_{57}}{\sigma_7} \\ \frac{\gamma_{16}\gamma_{56}}{\sigma_6} & \frac{\gamma_{26}\gamma_{56}}{\sigma_6} + \frac{\gamma_{27}\gamma_{57}}{\sigma_7} & \frac{\gamma_{36}\gamma_{56}}{\sigma_6} + \frac{\gamma_{37}\gamma_{57}}{\sigma_7} & \frac{\gamma_{46}\gamma_{56}}{\sigma_6} + \frac{\gamma_{47}\gamma_{57}}{\sigma_7} & * \end{bmatrix}$$

The three remaining diagonal entries are left as exercise for the reader (because they caused to matrix to run off the page). They are quite similar to the $\lambda_{22}$ entry. Now, the response matrix has the property that when you apply it to a vector of input potentials and each of the nodes, the resulting vector is the current response. That is, the entries is the resulting vector are the currents that flow out of each node. Notice that the current flowing out of node 1 due to a unit voltage at node 1 is negative because current would actually flow into node 1.

4.2. **Connections.** Before we can effectively talk about subdeterminants of the response matrix we must first introduce the idea of a connection. Given two sets of boundary nodes $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ we say $P$ and $Q$ are connected through the interior, or just "connected", if there is a permutation, $\tau$, of the nodes in $Q$ such that there a set of vertex disjoint paths $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ through the interior such that $\alpha_1$ joins $p_1$ to $q_{\tau(1)}$ and $\alpha_2$ joins $p_2$ to $q_{\tau(2)}$ and so on.

In other words, $P$ and $Q$ are connected if, after jumbling the nodes around however we like, we can connect every node in $P$ to one node in $Q$ and vice versa. Note that if the boundary nodes appear on a circle, and the network is drawn in the plane inside that circle, then there is only one permutation of the nodes that could possibly connect $P$ to $Q$.

The submatrix of the response matrix corresponding the connection $(P;Q)$ is the submatrix whose columns are nodes in $P$ and whose rows are nodes in $Q$. There is an example of this in the next section.

4.3. **Determinants.** The primary result that we will make use of is denoted Observation 3.3 on page 52 of [3].

**Claim 4.1.** If $\det\Lambda(P;Q) = 0$, then one or the other of the following two possibilities is true.

(1) There is no connection from $P$ to $Q$.
(2) There are (at least) two connections $\alpha$ and $\beta$ from $P$ to $Q$, with permutation $\tau_\alpha$ and $\tau_\beta$ of opposite sign.

We have called the above statement a claim. It is in fact a theorem whose complete proof can be found in [3], but the proof is complicated. There may be a more straightforward proof. We are interested in this paper in the application of the Star-K tool. We will see that we may be able to prove this assertion in general using Star-K's. We can certainly see that this is true in many examples using Star-K's. Consider the network shown in figure 6. We could use 4.1 to see that $det\Lambda(1,3;5,4) \neq 0$. Yet, suppose that we had never seen 4.1. How could we proceed then? Well, look at the sequence of star-K transformations that is shown. Since we know that the subdeterminant $det\Lambda(1,3;5,4)$ would equal zero if it had come from a star, those two extra non-zero edges push that determinant away from zero. This argument, while not complicated, is very exciting. Is it possible to prove claim 4.1 using star-K's? It hasn't been done yet, but it would be a very interesting project.

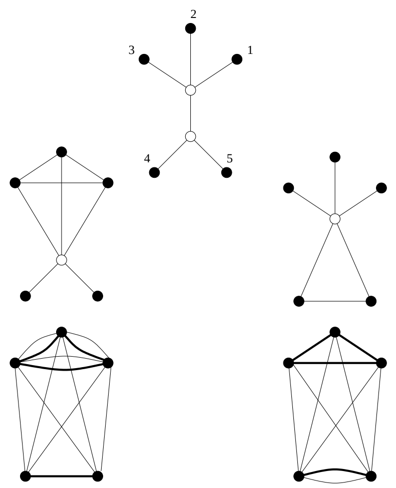


FIGURE 6. 5-node graph with certain two connections highlighted

## 5. N-gon in N-gon Graphs are Two to One.

5.1. **The Square in Square Graph.** The Square in Square graph is also $2-1$. We will show this by explicitly finding the coefficient of the $\alpha^2$ term and showing that it is also non-zero. By a very close analogy to the triangle-in-triangle calculation we can write down the terminating continued fraction version of the quadratic for the square-in-square easily. If we fix the parameter shown in Figure 5 then the quadratic in continued fraction form is

$$\lambda_{15} - \cfrac{\lambda_{14}\lambda_{58}}{\lambda_{48} - \cfrac{\lambda_{34}\lambda_{78}}{\lambda_{37} - \cfrac{\lambda_{23}\lambda_{67}}{\lambda_{26} - \cfrac{\lambda_{12}\lambda_{56}}{\alpha}}}} = \alpha. \tag{8}$$

When we clear denominators we can see the coefficient of the $\alpha^2$ term is

$$[\lambda_{48}\lambda_{23}\lambda_{67} + \lambda_{34}\lambda_{78}\lambda_{62} - \lambda_{48}\lambda_{37}\lambda_{62}]. \tag{9}$$

As in the triangle-in-triangle case, this is also a non-zero subdeterminant of entries in the response matrix. To see that it is so, note that D(2,4,7;3,6,8) $=$ $\lambda_{23}(\lambda_{46}\lambda_{78} - \lambda_{76}\lambda_{48}) + \lambda_{34}(\lambda_{26}\lambda_{78} - \lambda_{76}\lambda_{28}) + \lambda_{73}(\lambda_{26}\lambda_{78} - \lambda_{67}\lambda_{28})$ is equal to equation 9 if $\lambda_{46}$ and $\lambda_{28}$ are zero. They are zero in the response matrix of the

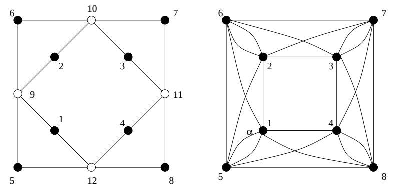


FIGURE 7. The Square in Square Graph and the Star-K Equivalent Graph

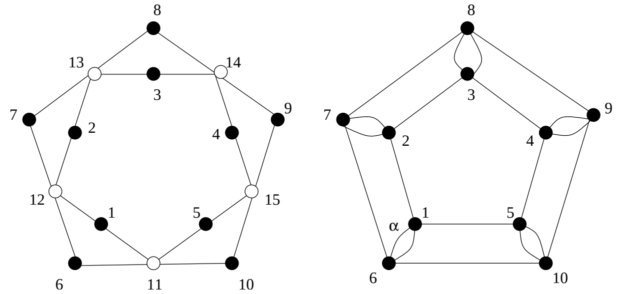


FIGURE 8. The Pentagon in Pentagon Graph and the Star-K Equivalent Graph

square-in-square because those pairs of nodes are not connected through the interior. Refer to claim 4.1; there is only one way to make the connection (2,4,7;3,6,8) so this determinant is non-zero. At this point we could speculate that the coefficient of the $\alpha^2$ term in the pentagon-in-pentagon graph shown in Figure 6 will be D(2,8,4,10;7,3,9,5).

5.2. **The Pentagon-in-Pentagon Graph.** The pentagon-in-pentagon graph shown in Figure 7 has an associated quadratic that can be easily written down in its terminating continued fraction form as

$$\lambda_{16} - \cfrac{\lambda_{15}\lambda_{6,10}}{\lambda_{5,10} - \cfrac{\lambda_{45}\lambda_{9,10}}{\lambda_{49} - \cfrac{\lambda_{34}\lambda_{89}}{\lambda_{38} - \cfrac{\lambda_{23}\lambda_{78}}{\lambda_{27} - \cfrac{\lambda_{12}\lambda_{67}}{\alpha}}}}} = \alpha. \tag{10}$$

Again, we can clear denominators and this will show the coefficient of the $\alpha^2$ term to be

$$[\lambda_{5,10}\lambda_{49}\lambda_{38}\lambda_{27} + \lambda_{5,10}\lambda_{34}\lambda_{89}\lambda_{27} - \lambda_{5,10}\lambda_{49}\lambda_{23}\lambda_{78} - \lambda_{38}\lambda_{45}\lambda_{9,10}\lambda_{27} + \lambda_{45}\lambda_{9,10}\lambda_{23}\lambda_{78}].$$

That is, in fact, D(2,8,4,10;7,3,9,5) because of the zeros in the response matrix. There is only one way to make this connection as before which guarantees we do in fact have a quadratic. This leads us to make the following conjecture. However, since we prove it in the next section we had better label it a theorem.

**Theorem 5.1.** *If you number an n-gon-in-n-gon graph clockwise around the inside then clockwise around the outside from the same starting side, then assign a parameter to one of the edges joining nodes 1 and n+1, then*

$$D(2, n+3, 4, n+5, ..., n-1, 2n; n+2, 3, n+4, 5, ..., 2n-1, n)$$

*is the coefficient of the quadratic term.*

5.3. **Proof of Theorem.** This section outlines a recurrence relation for the coefficients of a Linear Fractional Transformation which is equivalent to our terminating continued fraction.

Chrystal's book, [4], supplied the ideas for this section, but we have interpreted the results in a new and interesting way. First, we write down our notation for terminating continued fractions

$$\frac{p_n}{q_n} = a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \cfrac{b_5}{a_5}}}}. \tag{11}$$

Chrystal shows that $p_n$ and $q_n$ can be defined recursively by identical formulas; $p_n$ and $q_n$ differ only because of their initial conditions: $p_0 = 1, p_1 = a_1; q_1 = 1, q_2 = a_2$. The formulas are

$$p_n = a_n p_{n-1} + b_n p_{n-2} \quad \text{and,} \tag{12}$$
$$q_n = a_n q_{n-1} + b_n q_{n-2}. \tag{13}$$

These recurrence relations are easy to prove by induction. First, the $p_0, p_1, q_1$ and $q_2$ cases are trivial, then, if we assume we have $p_{n-1}$ and $q_{n-1}$ terms, we can find the $p_n$ and $q_n$ by simply replacing the term $a_{n-1}$ by $a_{n-1} + \frac{b_n}{a_n}$.

Consider

$$\frac{p_{n-1}}{q_{n-1}} = \frac{a_{n-1}p_{n-2} + b_{n-1}p_{n-3}}{a_{n-1}q_{n-2} + b_{n-1}q_{n-3}} \quad \text{goes to} \quad \frac{(a_{n-1} + \frac{a_n}{b_n})p_{n-2} + b_{n-1}p_{n-3}}{(a_{n-1} + \frac{a_n}{b_n})q_{n-2} + b_{n-1}q_{n-3}}.$$

Now,

$$\frac{(a_{n-1} + \frac{a_n}{b_n})p_{n-2} + b_{n-1}p_{n-3}}{(a_{n-1} + \frac{a_n}{b_n})q_{n-2} + b_{n-1}q_{n-3}} = \frac{a_n(a_{n-1}p_{n-2} + b_{n-1}p_{n-3}) + b_n p_{n-2}}{a_n(a_{n-1}q_{n-2} + b_{n-1}q_{n-3}) + b_n q_{n-2}} = \frac{a_n p_{n-1} + b_n p_{n-2}}{a_n q_{n-1} + b_n q_{n-2}}.$$

When we compare (11) and (12) we find that our $\alpha$ takes the place of the $a_n$. Since $a_n$ only appears in the $p_n th$ and $q_n th$ terms, and in equation (11) we have $\frac{p_n}{q_n} = \alpha$ we can write our terminating continued fraction in the form of a linear fractional transformation (LFT). This LFT is

$$\alpha = \frac{\alpha p_{n-1} + b_n p_{n-2}}{\alpha q_{n-1} + b_n q_{n-2}}. \tag{14}$$

In this form we can easily write down the quadratic that corresponds to this LFT.

$$\alpha^2 q_{n-1} + \alpha(b_n q_{n-2} - p_{n-1}) - b_n p_{n-2} = 0 \tag{15}$$

Remarkably, the coefficient of $\alpha^2$ is simply $q_{n-1}$. The discriminant, which we will also want to examine is

$$(b_n q_{n-1} - p_{n-1})^2 + 4b_n p_{n-2} q_{n-1}. \tag{16}$$

We can also use the work outlined in Chrystal to equate $p_n$ and $q_n$ to certain determinants. In fact, it is not hard to see how these recurrence relations define certain subdeterminant of the response matrix. For an example consider the following determinant.

$$\begin{vmatrix} \lambda_{48} & \lambda_{34} & 0 \\ \lambda_{78} & \lambda_{37} & \lambda_{67} \\ 0 & \lambda_{23} & \lambda_{26} \end{vmatrix}$$

This is the coefficient of $\alpha^2$ for the square-in-square graph. This determinant, because $\lambda_{28}$ and $\lambda_{64}$ are zero, corresponds to the connection $(8,3,6;4,7,2)$ which can also be written as the connection (2,7,4;6,3,8). This connection can only be made one way in the original graph, so this determinant is non-zero. This is the same conclusion as we reached by hand and showed in equation (9), but at that point it was just coincidence. Now we can write out the coefficient of $\alpha^2$ for an arbitrarily n-gon in n-gon graph:

$$\begin{vmatrix} \lambda_{2,n+2} & \lambda_{2,3} & & 0 & 0 & \dots & 0 & 0 \\ \lambda_{n+2,n+3} & \lambda_{3,n+3} & \dots & 0 & 0 & & 0 & 0 \\ & \vdots & \ddots & & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & \lambda_{i,n+i} & \lambda_{i,i+1} & & 0 & 0 \\ 0 & 0 & & \lambda_{n+i,n+i+1} & \lambda_{i+1,n+i+1} & \dots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \ddots & \vdots & \\ 0 & 0 & & 0 & 0 & \dots & \lambda_{n-1,2n-1} & \lambda_{n-1,n} \\ 0 & 0 & \dots & 0 & 0 & 0 & \lambda_{2n-1,2n} & \lambda_{n,2n} \end{vmatrix}.$$

This determinant corresponds to the connection

$$(2, n+3, 4, n+5, ..., n-1, 2n; n+2, 3, n+4, 5, ..., 2n-1, n),$$

which we know to be non-zero. We know this because this is the only permutation of the nodes that allows for vertex disjoint paths through the interior from P to Q, as in claim 4.1. This guarantees that we have a genuine quadratic term in every case. It remains only to be shown that the discriminant is positive and not always zero to show that we sometimes have two real, positive solutions.

A topic for future study, besides examining the discriminant, would be to take the limit as n goes to infinity in the matrix above and see if you can come up with a meaning associated with the result. This "circle-in-circle" graph may have interesting properties.

## 6. Challenge: A Recoverable Flower

A recoverable flower may be found with the star-K tool. By flower, we mean to say a graph with no boundary spikes, and no boundary to boundary connections. I propose as a challenge that someone find a recoverable flower, or prove that no such can exist. We once thought Figure 9 could be shown, using star-K transformations, to be recoverable. Even though this no longer appears true, it is still pretty.

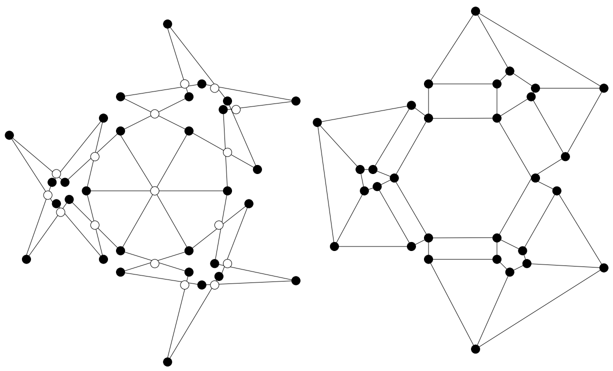

Figure 9. A Flower and Its Star-K Equivalent

## 7. The Genus of Graph

It is also possible to conclude, based on the connections that can not fit in the plane, that certain response matrices must have come from graphs that can only be embedded on more exotic surfaces. For example, consider a four node network whose boundary nodes are on a circle numbered 1,2,3,4 in counterclockwise order. Then, if the network is planar, it impossible to connect node 1 to node 3 and simultaneously connect node 2 to node 4 via disjoint paths. Thus, given a response matrix and a circular ordering of the boundary nodes, if $det\Lambda(1,2;3,4) \neq 0$ then the network is not planar! This easily generalizes through increasingly complex "cross wirings" to conclusions about certain networks not fitting on the sphere. Eventually it would ideal to pin down a definition of the genus of a network, and determine the genus of the a network from its response matrix.

## 8. Some Other Cool Examples

8.1. **Recovering the Tower of Hanoi Network With Star-K's.** We prove that the Tower of Hanoi network shown in figure 10 is recoverable by examining a sequence of pictures that represent star-K transformations of each interior node. We note that each step is reversible using the quadrilateral relation, and we have proven that this is so.

We are left with an interesting point to ponder. Is it possible to have a sequence of star-K networks where one of the steps is not reversible, but the graph itself remains recoverable?

Here is another interesting question to ponder. Is it possible to write down all recoverable 5-boundary node graphs? What about 4-boundary node graphs?

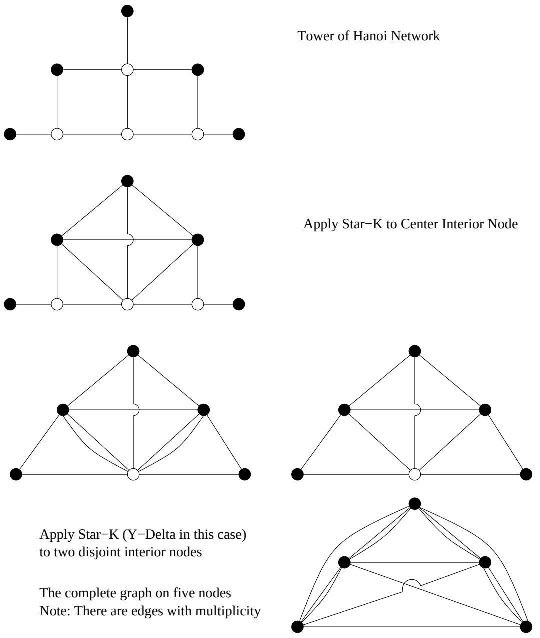


FIGURE 10. Recovering the Tower of Hanoi Graph

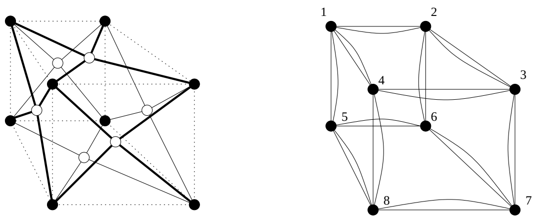


FIGURE 11. A Cube of Plus-Graphs and Its Star-K Equivalent

8.2. **The Cube is** $8-1$**.**

**Claim 8.1.** The cube shown in figure 11 is $8-1$.

This argument will follow from the square-in-square graph being $2-1$, and this graph containing three functionally disjoint square-in-square graphs, thus making it $2^3-1$. However, before we can make such and argument we must first give some explanation of the diagram.

The left diagram shows six plus-graphs embedded on the cube such that each one takes up a side. The vertices of the cube are the boundary nodes. The Star-K equivalent is drawn with all the diagonals suppressed to make the picture comprehensible, the existence of these diagonals are however, essential.

8.3. **The Race Track Graph.** The race track graph also has an associated equation that looks quadratic, but more analysis is needed to show which terms do or do not vanish. Many other seemingly $2-1$ graphs can be constructed by joining plus-graphs together into various chains that loop back on themselves. The method of examining the terminating continued fraction form of the resultant quadratic equation may prove useful in these case as well. The difference is that in each case we get a product of continued fractions, or some more complicated behavior.

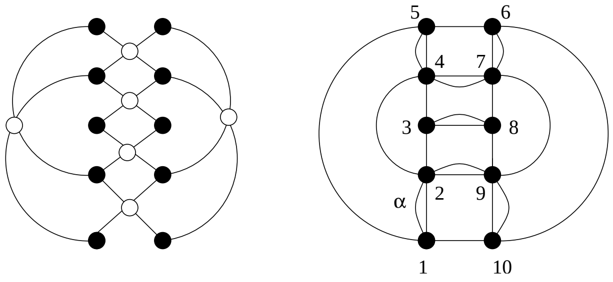


FIGURE 12. The Race Track Graph and Its Star-K Equivalent

8.4. **Circles and Rays.** In this section the goal is to produce a straightforward exposition of the network with two circles and three rays, and also the network with two circles and four rays. We have drawn a sequence of Star-K transformations on both graphs that could serve as a guide. The claim in [2] is the network with two circles and three rays is not recoverable and the network with two circles and four rays is recoverable. We should be able to prove or disprove this claim easily, but it has not yet been done.

## References

[1] Jeffrey Russell. Solving the Inverse Problem. 2003.
[2] Ernie (John) Esser. On Annular Graphs. 2000.
[3] jim Curtis, B., and James A. Morrow. "Inverse Problems for Electrical Networks." Series on applied mathematics – Vol. 13. World Scientific, ©2000.
[4] Chrystal, George. Algebra, an Elementary Text-Book for The Higher Classes of Secondary Schools and for Colleges. New York, Chelsea, 1959 Volume II, pages 491-502

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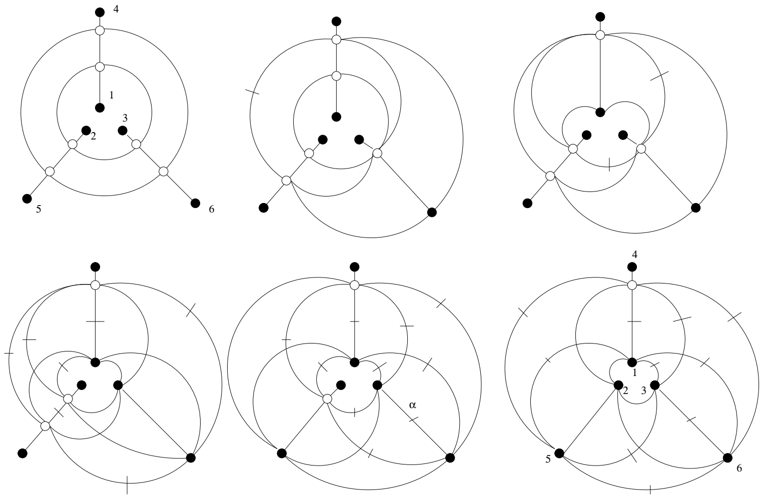


Figure 13. The 2 Circle 3 Ray Graph and Its Star-K Equivalent

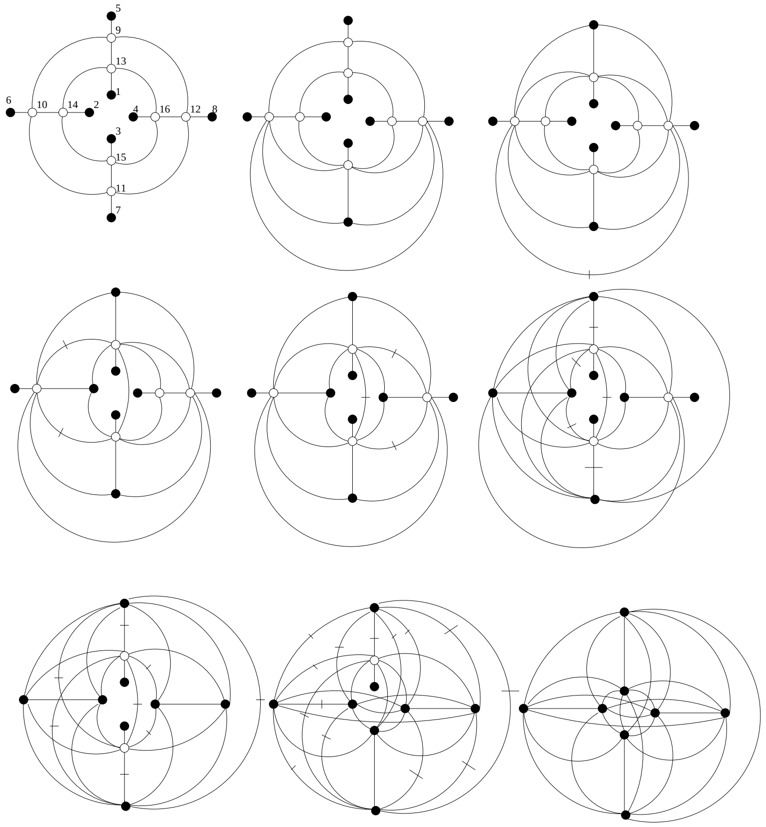


Figure 14. The 2 Circle 4 Ray Graph and Its Star-K Equivalent