ON AN INVERSE BOUNDARY VALUE PROBLEM

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In this note we shall diseuss the following problem. Let D be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with Lipschitzian boundary ∂D, and γ be a real bounded measurable function in D with a positive bower bound. Consider the differential operator

$$L_\gamma(W) = \nabla.(\gamma \nabla W)$$

acting on functions of $H^1(D)$ and the quadratic form $Q_\gamma(\phi)$, where the functions ϕ are restrictions to ∂D of functions in $H^1(\mathbb{R}^n)$, defined by

$$Q_\gamma(\phi) = \int_D \gamma(\nabla W)^2 dx \quad , W \in H^1(\mathbb{R}^n),\ W\Big|_{\partial D} = \phi$$

$$L_\gamma W = \nabla.(\gamma \nabla W) = 0 \quad \text{in D.}$$

The problem is then to decide whether γ is uniquely determined by Q_γ and to calculate γ in terms Q_γ, if γ is indeed determined by Q_γ.

This problems originates in the following problem of electrical prospection. If D represents an in-homogeneous conducting body with electrical conductivity γ, determine γ by means of direct current steady state electrical measurements carried aut on the surface of D, that is, without penetrating D. In this physicak situation $Q_\gamma(\phi)$ represents the power necessary to maintain an electrical potential γ on ∂D.

In principle Q_γ can be determined through measurements effected on ∂D and contains all the information about γ which can be thus obtained.

But let us retwen to our mathematical problem. Let on ∂D and in the space of quadratic forms $Q(\phi)$

$$\|\phi\|^2 = \int_D |\nabla u|^2 dx \quad ; \; u\big|_{\partial D} = \phi \;, \; \Delta u = 0 \quad \text{in } D.$$

$$\|Q\| = \sup_{\|\phi\| \le 1} |Q(\phi)|$$

Then the mapping

$$\Phi : \gamma \to Q_\gamma.$$

is bounded and analytic in the subset of $L^\infty(D)$ consisting of functions which are real and have a positive lower bound. Our goal is then to determine whether Φ is inyective, and invert Φ if this is the case. This we are yet unable to do, and is, as far as we known, an open problem. However we shall show that $d\Phi|_{\gamma = \text{const.}}$ is indeed inyective , that is, the linearized problem has an affirmative answer. If $d\Phi|_{\gamma = \text{const}}$, which is a linear operator, had a closed range, one could conclude that Φ itself is inyective in a sufficiently small neighborhood of γ= const. But the range of $d\Phi$ is not closed and the desired conclusion cannot be obtained in this fashion. Nevertheless, as we shall see below, if γ is sufficiently close to a constant , it is nearly determined by Q_γ and in some cases it can be calculated with an error much smaller than $\|\gamma - \text{const}\|_{L^\infty}$

To show this let us first obtain an expression for the solution of the equation

$$L_\gamma(W) = \nabla.(\gamma\nabla W) = 0 \;, \; W\big|_{\partial D} = \phi. \quad \gamma = 1 + \delta$$

Let $W = u + v$, where $\Delta u = L_1 u = 0$, $u\big|_{\partial D} = \phi$. Then

$\delta = 0 \Rightarrow v = 0$

$$.L_\gamma(W) = L_{1+\delta}(u+v) = L_1 v + L_\delta v + L_\delta u = 0$$

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Since $u\big|_{\partial D} = W\big|_{\partial D}$ we have $v\big|_{\partial D} = 0$ and $v \in H_0^1(D)$, the closure an operator from $H_0^1(D)$ into $H^{-1}(D)$, has a bounded inverse G, and from the last expression we obtain

$$v + GL_\delta v = -GL_\delta u.$$

and

(1) $$v = -\left[\sum_0^\infty (-1)^j (GL_\delta)^j\right](GL_\delta u) \qquad (D) \quad \delta$$

Since for $W \in H_0^1(D)$, $\| L_\delta W\|_{H^{-1}(D)} \le \|\delta\|_{L^\infty} \|W\|_{H_0^1(D)}$ if A denotes the norm of G, the series above will converge for $\|\delta\|_{L^\infty} A < 1$ and

(2) $$\|v\|_{H^1(D)} \le \frac{A\|\delta\|_{L^\infty}\|\phi\|}{1 - A\|\delta\|_{L^\infty}}$$

?

From (1) it follows that ϕ is analytic at $\gamma = 1$. The same argument would show that ϕ is analytic at any other point γ.

Next let us calculate $d\phi|_{\gamma=1}$. We have

$v|_{\partial D} = 0$

(3) $$Q_{1+\delta}(\phi) = \int_D (1+\delta)|\nabla w|^2 dx = \int_D \left[(1+\delta)|\nabla u|^2 + 2(\nabla u.\nabla v) + 2\delta(\nabla u.\nabla v) + (1+\delta)|\nabla v|^2\right] dx$$

The contribution of the second term in the integrand of the last integral vanishes on account of the fact that $\Delta u = 0$. Furthermore, from (1) one sees readily that the parts linear in δ of the last two terms in the integrand vanish. Thus setting $d\gamma = \delta$ we obtain

$$dQ_\gamma(\phi)|_{\gamma=1}(\delta) = \int_D \delta |\nabla u|^2 dx \quad , \ \Delta u = 0 \quad , \ u|_{\partial D} = \phi$$

To show that $dQ_\gamma|_{\gamma=1}$ is inyective, we merely have to show that if the last integral vanishes for all u with $\Delta u = 0$ then $\delta = 0$

$$(4) \qquad \int_D \delta(\nabla u_1 . \nabla u_2)dx = 0$$

whenever $\Delta u_1 = \Delta u_2 = 0$ in D. Now let Z be any vector in $\mathbb{R}^n$ and $\underline{a}$ another vector such that $|\underline{a}| = |Z|$, $\underline{a}.Z) = 0$. Then the functions

$$(5) \qquad u_1(x) = e^{\pi i(Z.x)+\pi(\underline{a}.x)}, \; u_2 = e^{\pi i(Z.x)-\pi(\underline{a}.x)}.$$

are harmonic, and substituting in (3) we obtain

$$2\pi|Z|^2 \int_D \delta(x)e^{2\pi i(Z.x)}dx = 0 \quad , \; \forall Z$$

whence it follows that $\delta = 0$.

Now let us return to $Q_\gamma(W)$. We set again $\gamma = 1+\delta$ and introduce the bilinear form

$$B(\phi_1,\phi_2) = \frac{1}{2}\Big[Q_\gamma(W_1+W_2) - Q_\gamma(W_1) - Q_\gamma(W_2) \Big]$$

and setting $W_j = u_j+v_j$, $j = 1,2$, $\Delta u_j = 0$, $u_j|_{dD} = \phi_j$ we obtain

$$B(\phi_1,\phi_2) = \int_D (1+\delta)(\nabla u_1 . \nabla u_2) + \delta\big[(\nabla u_1 . \nabla v_2)+(\nabla u_2 . \nabla v_1)\big] +$$

$$+ (1+\delta)(\nabla v_1 . \nabla v_2)\Big]dx.$$

Now, substitution of the exponentials in (5) for u_1 and u_2 in the preceding expression (taking $\underline{a}$ to be a function of Z such that $|a| = |Z|$, $(\underline{a}.Z) = 0$) yields

$$(6) \qquad \hat{\gamma}(Z) = \hat{F}(Z) + R(Z)$$

where $\hat{\gamma}(Z)$ is the Fourier transform of γ extended to be zero outside D, the function

$$\hat{F}(Z) = \frac{1}{2\pi^2|Z|^2} B(e^{i\pi(Z.x)+\pi(\underline{a}.x)}, \; e^{i\pi(Z.x)-\pi(\underline{a}.x)})$$

is known and, as follows readily from(2),

$$(7) \qquad |R(Z)| \leq C\|\delta\|^2_{L^\infty}\, e^{2\pi r|Z|}$$

provided that $A\|\delta\|_{L^\infty} \leq 1 - \varepsilon$, where C depends only on D and ε, and r is the radius of the smallest sphere containing D. Now $R(Z)$ is too large to permit estimating $\gamma(Z)$. However, under favorable circumstances it is still possible to obtain satisfactory information about γ. Choose α, $1 < \alpha < 2$, then for

$$(8) \qquad |Z| \leq \frac{2-\alpha}{\pi\gamma} \log \frac{1}{\|\delta\|_{L^\infty}} = \sigma$$

we have $|R(Z)| \leq C\|\delta\|^{\alpha}_{L^\infty}$. Let η be a function such that $\hat{\eta} \in C^\infty$, supp $\hat{\eta} \subset \{|Z| \leq 1\}$ $\hat{\eta}(0) = 1$, and let $\eta_\sigma(Z) = \sigma^n \eta(\sigma Z)$. Then we have

$$\hat{\gamma}(Z)\ \hat{\eta}(\tfrac{Z}{\sigma}) = \hat{F}(Z)\hat{\eta}(\tfrac{Z}{\sigma}) + R(Z)\hat{\eta}(\tfrac{Z}{\sigma})$$

and

$$(9) \qquad (\gamma * \eta_\sigma)(x) = (F * \eta_\sigma)(x) + \rho(x)$$

where $*$ denotes convolution and

$$|\rho(Z)| \leq C\|\delta\|^{\alpha}_{L^\infty}\int |\hat{\eta}(\tfrac{Z}{\sigma})|\, dZ =$$

$$= C_1\|\delta\|^{\alpha}_{L^\infty}\left[\log \frac{1}{\|\delta\|_{L^\infty}}\right]^n$$

where C_1 depends only on D, α and ε.

Thus if $\|\delta\|_{L^\infty}$ is sufficiently small, (9) gives an approximation for $\gamma * \eta_\sigma$ with an erroe which is much smaller than $\|\delta\|_{L^\infty}$. Clearly, if $\|\delta\|_{L^\infty}$ is small then σ is large and $\gamma * \eta_\sigma$ is itself, in some sense, a good approximation to γ.

Approximations to the funcion γ itself be obtained if one assumes that γ, extended to be equal to 1 outside D, is

$$\hat{\delta}(Z) = \hat{F}_1(Z) + R(Z)$$

where F_1 is known and $R(Z)$ is the same as in (6). One then calculates $\delta(x)$ by integrating over $|Z| \leq \sigma$ with σ as in(8) and estimates the error by using the decay of $\hat{\delta}$ at ∞. Thus one obtains

$$\gamma(x) = F_2(x) + \rho(x)$$

where $F_2(x)$ is known and.

$$|\rho(x)| \leq C\|\delta\|^{\alpha}_{L^\infty}\left[\log \frac{1}{\|\delta\|_{L^\infty}}\right]^{n} + CM\left[\log \frac{1}{\|\delta\|_{L^\infty}}\right]^{-m+n}$$

where M is a bound for the derivatives of order m of γ.

Bibliography

We have been unable to find treatments of the problem discussed above in the literature, at least not in the general setting in which we are interested. Similar problems have been studied in the papers listed below.

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